WORKING PAPERS

N° TSE-571

April 2015

“More Haste, Less Speed? Signaling through Investment Timing”

Catherine Bobtcheff and Raphaël Levy

Toulouse School of Economics

More Haste, Less Speed? Signaling through Investment Timing*

Catherine Bobtcheff[†] Raphaël Levy[‡]

April 24, 2015

Abstract

We consider a real option model in which a cash-constrained entrepreneur learns prior to investing, but at a speed which is private information. The entrepreneur seeks outside funding, and uses the timing of his investment to signal his confidence in the venture, and accordingly obtain cheaper credit. In the benchmark case with no informational friction, we show that the optimal investment date may be non-monotonic or decreasing in the learning speed, depending on the prior NPV of the project: better learning increases the value of the option to wait, but also increases the speed of updating. In the presence of asymmetric information, the cash constraint may result in distortions in investment timing, and the inefficiency is higher the more stringent the cash shortage. Inefficient investment policy may take both the form of hurried investment (as compared to the benchmark), when both entrepreneur types learn sufficiently fast, and of delayed investment, when the slow-learning type does not learn fast enough. Therefore, the severity of the cash constraint affects the magnitude of the timing distortion, but not its direction.

Keywords: Signaling, investment timing, financing of innovation, real options.

*We thank Bruno Biais, Mathias Dewatripont, Denis Gromb, Johannes Hörner, Thomas Mariotti, David Martimort, Sophie Moinas, Andrea Prat, and Jean Tirole for very valuable feedback. We also thank seminar audiences at CREST-LEI, the European University Institute, Toulouse School of Economics, University of Copenhagen, University of Mannheim, University of Lausanne, as well as participants at the ASSET, SAET and ESEM conferences. C. Bobtcheff received funding from the European Union's Seventh Framework Programme (FP7/2007-2013) under the grant agreement 230589. R. Levy received financial support from the French National Research Agency (A.N.R. project "Regulate to Promote Information Sharing").

†Toulouse School of Economics (CNRS-LERNA)

‡Mannheim University

1 Introduction

Innovation is a major driver of economic growth, and has accordingly been of central interest for scholars, policy-makers, and practitioners. In particular, much attention has been devoted to the question of the financing of innovation. An important concern is notably that financial frictions might lead to underinvestment in R&D: the literature has indeed documented a wedge between the rate of return required by an entrepreneur to launch a R&D project and the rate of return of capital required by external investors to finance this project (Hall and Lerner, 2010; Kerr and Nanda, 2014). One of the main reasons explaining such a wedge has to do with asymmetric information: the innovator often has superior information than potential financiers about, e.g., his talent, the quality of his project, or the effort he puts in to run his business. Information and financial frictions are indeed particularly relevant to innovative firms: first, they are often run by young entrepreneurs with no established reputation, so that information problems are more severe; second, a significant chunk of the innovation potential comes from small firms with little or no cash. In this paper, we underline how information and financial frictions impact entrepreneurs by shaping their incentives to learn. Actually, experimentation is a critical dimension of the innovation process: in the face of uncertainty, entrepreneurs first need to run tests so as to learn whether further investments are worthwhile. In a recent paper, Ewens *et al.* (2014) show that the recent fall in experimentation costs (cloud computing, accelerators...) has reduced financing constraints for projects with the greatest option value, and has mostly benefitted to young and inexperienced entrepreneurs. This suggests that information and financial frictions most hit small firms for which efficient learning is critical to the success of the venture. In line with this idea, we show how asymmetric information may provide incentives to under-experiment or over-experiment in the presence of a cash constraint, raising the concern that innovation could occur too early or too late as compared to the efficient investment policy.

We consider a continuous time model which combines the following ingredients: (a) an entrepreneur owns an irreversible project over which he learns as long as the project has not been launched (the entrepreneur holds a real option); (b) his learning ability, hence the value of the option, is private information;[1] (c) the entrepreneur is cash-constrained

[1] In the paper, we will interchangeably use the terms learning ability and learning speed to refer to the quality of the entrepreneur's learning technology. Formally, this will be captured by a single variable measuring the precision of the signal he observes.

and needs outside funding to finance the project. Private information on learning ability implies that the entrepreneur has superior information about the value of the project compared to the financier, hence a signaling problem: the date at which the entrepreneur launches the project conveys information on how confident he is, and the entrepreneur would have the bank believe that he is as confident as possible in order to obtain cheaper credit.

To better understand the signaling content of the investment timing, we first analyze the benchmark case with complete information, and examine how the investment timing varies with the entrepreneur's learning ability. The learning ability here plays a dual role: on the one hand, a higher ability increases the value of the option to wait, which raises the minimal level of confidence on the project the entrepreneur wants to reach before investing; on the other hand, learning is faster, and any such cutoff level of confidence is reached earlier. When the prior net present value of the project is negative, the investment date is always decreasing in the learning ability; however, when it is nonnegative, the investment date becomes non-monotonic: both fast and slow learners invest early, the former because they get confident about the project very fast, the latter because the option to wait has little value, and there is therefore no point waiting. This suggests that the signaling content of the investment timing is intrinsically ambiguous: if the entrepreneur wants to signal a high learning ability, it is a priori unclear whether he should invest early or late.

Under private information on learning ability, we first establish that the entrepreneur can always reach his complete information payoff whenever he holds sufficient cash: when a high share of the investment is internally financed, a slow-learning entrepreneur is unwilling to distort his investment timing to pretend he learns fast, as the cost of inefficient investment policy dwarfs the benefit of cheaper credit. However, the investment timing has to be distorted whenever the entrepreneur holds insufficient cash, and the distortion is more severe the higher the cash shortage. This is because the signaling concerns become more salient when a higher share of the project is financed with outside funds. Interestingly, though, while the magnitude of the distortion depends on the severity of the cash constraint, the direction of the distortion is orthogonal to the entrepreneur's net worth, and only depends on how the option values of each type, hence their learning speeds, compare. When the slow-learning type learns fast enough, the equilibrium involves hurried investment, while it involves delayed investment in case the slow-learning type learns sufficiently slowly. The intuition has to do with the difference in the dynamics of beliefs for entrepreneurs learning at different speeds. At the outset, both types share the same prior belief on the project; as time goes by, the fast-learning type becomes increasingly more optimistic than the slow-learning type in a first phase; but, because learning exhibits decreasing returns, this phase is followed by a phase in which the slow-learning type catches up on beliefs. Separation is possible when the difference between each type's confidence in the project is high, that is, at a date when the good type has accumulated enough superior information as compared to the bad type. When the slow-learning type learns sufficiently fast (resp. slowly), the phase in which the difference in beliefs expands is relatively short (resp. long), and separation is achieved by investing earlier (resp. later). This result that the distortion possibly goes in both directions has to do with the impossibility to rank types according to their intrinsic preference over investment timing. Put differently, the single-crossing property does not hold, which implies that early investment may be relatively preferred by a fast-learning or by a slow-learning entrepreneur depending on how their learning speeds compare. Accordingly, there is no systematic relationship between the ordering of investment dates under complete information and the direction of the distortion under asymmetric information: whenever the good type invests later than the bad type under complete information, he must invest even later under asymmetric information; however, in the opposite case where the good type invests earlier under complete information, the distortion may involve either hurried or delayed investment. Relatedly, asymmetric information may give rise to two kinds of reversals: (a) a reversal of investment timings, in that the good type may invest later than the bad type, although he would invest earlier under perfect information; (b) the good type may be less confident than the bad type upon investing, which never happens under complete information.

In terms of empirical predictions, our results suggest a relationship between the nature of the distortion in the investment policy and the speed of learning in the market: in industries characterized by fast learning (i.e., where even slow learners learn sufficiently fast), one should observe over-investment and under-experimentation, that is, an inefficiently high failure rate conditional on investment. Conversely, we should expect under-investment and over-experimentation in industries where learning is slower. This is consistent with evidence on the pharmaceutical industry, where learning is typically considered slow: for instance, Guedj and Scharfstein (2004) focus on drug development and establish that cash-constrained firms tend to invest less than unconstrained ones, and have a lower failure rate. This is also consistent with Henkel and Jell (2010), who establish that long patent examination deferments are particularly frequent in the pharmaceutical and chemical industries.

Our paper relates to the literature on real options (Dixit and Pindyck, 1994), and on exponential/Poisson learning, notably Keller *et al.* (2005) and Décamps and Mariotti (2004). It is more particularly related to a strand of the recent literature on experimentation dealing with the impact of asymmetric information. Agency problems may involve adverse selection (private learning), moral hazard (unobservable learning effort), or both. Several papers have considered the question of the design of the optimal contract (or the optimal mechanism) for experimenting agents, for instance Manso (2011), Gerardi and Maestri (2012), or Halac *et al.* (2013, 2014). In particular, a series of papers has focused on the question of the financing of experimentation (Bergemann and Hege, 1998, 2005; Hörner and Samuelson, 2013; Bouvard, 2014; Drugov and Macchiavello, 2014). A distinct class of papers have analyzed models with no commitment where the investment timing can be used as a signaling device (Grenadier and Malenko, 2011; Morellec and Schürhoff, 2011; Bustamante, 2012). Our paper, which also features a "real option signaling game", methodologically belongs to this stream of papers.

However, most of the aforementioned papers model private information on a variable which monotonically affects the optimal timing decision (e.g, the prior belief, or the cost of investment).[2] In turn, this monotonicity implies that the distortion always goes in one direction. For instance, Morellec and Schürhoff (2011) and Bustamante (2012) derive that investment is hurried in equilibrium, while it is delayed in Bouvard (2014). Grenadier and Malenko (2011) consider four different corporate finance applications, and establish that the direction of the distortion depends on the application one considers (more precisely, on what the sender wants to signal), but is always the same within one application.[3] Instead, we model private information on the parameter which directly measures the quality of learning, which notably implies a non-monotonic relationship between the optimal investment date and the learning speed. In such a context, we establish that signaling a

[2] A notable exception is Halac *et al.* (2013), who consider a setup with private information on the probability of success of the project, in which learning comes from observing past successes or failures. Our two models accordingly bear some formal similarities.

[3] Relatedly, Bebchuk and Stole (1993) show that informational frictions may lead to overinvestment or underinvestment according to the way the information asymmetry is modeled.

high learning speed may involve either hurried or delayed investment.[4]

The paper is organized as follows: Section 2 is devoted to the presentation of the model. In Section 3, we characterize the optimal investment timing policy under perfect information. In Section 4, we derive the unique equilibrium of the signaling game, and discuss how it compares with the solution under complete information. Section 5 reviews some empirical implications of the model. In Section 6, we discuss the robustness of the results. Finally, Section 7 concludes. All the proofs are relegated to the Appendix.

2 The model

A risk neutral entrepreneur owns a project with *ex ante* uncertain value. With probability p_0, the project is of high quality, and yields a revenue $R > 0$. With probability $1 - p_0$, the project is of low quality, and yields zero revenue. Investment involves an irreversible cost $I \in (0, R)$. The entrepreneur is risk neutral and discounts future revenues and costs at rate $r > 0$.

The entrepreneur initially holds an amount of cash $A < I$, which is continuously capitalized at rate r.[5] Therefore, the entrepreneur needs to raise outside funds from competitive investors if he wants to launch the project before a date $\widetilde{t}$ such that $Ae^{r\widetilde{t}} = I$.

2.1 Learning environment

The entrepreneur decides the date $t \geq 0$ at which investment is triggered, if at all. The rationale behind waiting is that the entrepreneur learns about the quality of the project as long as investment has not taken place. We assume, following Décamps and Mariotti (2004), that he observes a signal modeled as a Poisson process with intensity $\lambda > 0$ if the project is of low quality, and with intensity 0 in case the project is of high quality. Therefore, "no news is good news": a jump perfectly identifies a low-quality project, whereas the entrepreneur gets increasingly optimistic about project quality as long as nothing is observed.[6] The signal is observed for free, so that the benefit of learning on

[4]Notice that Halac *et al.* (2013) derive a non-monotonicity result similar to ours under complete information, but find that asymmetric information always results in under-experimentation (hurried investment.

[5]This is without loss of generality. We later argue that we could allow the risk-free interest rate faced by the entrepreneur to differ from the discount rate. See Section 6.1 for more details.

[6]One may think of the pre-investment or learning period as a phase during which the entrepreneur runs tests on his project, for instance the phase I of the FDA's drug review process. The assumption that

the project is only traded against the cost of delaying investment.[7] Let $s(\lambda, t)$ denote the probability of receiving no signal before date t :

$$s(\lambda, t) = p_0 + (1 - p_0)e^{-\lambda t}. \tag{1}$$

Using Bayes' rule, the entrepreneur's beliefs on the project conditional on no signal being observed before date t, read

$$p^*(\lambda, t) = \frac{p_0}{s(\lambda, t)} = \frac{p_0}{p_0 + (1 - p_0)e^{-\lambda t}}. \tag{2}$$

From (2), one sees that the speed of updating increases with λ : a higher λ implies that no news is better news. In the analysis, we will propose a dual interpretation of λ : on the one hand, it captures the specific efficacy/precision of the testing technology, which is privately known to the entrepreneur; on the other hand, λ also captures the speed of learning in a given industry/market. The difference in learning speeds across industries could stem for instance from technological reasons (some industries, e.g. biotechnologies, pharmaceuticals, are intrinsically more difficult, in that a wider variety of tests has to be run, as compared to "easier" industries, like software). To avoid notation inflation, we stick to one single notation λ, but implicitly have in mind that λ is composed of a privately observed component (entrepreneur-specific talent) and a publicly observed component (industry-specific speed of learning).[8] This will allow us to make different predictions on the impact of the information and financial frictions in different markets.

2.2 The loan contract

The entrepreneur decides to wait and learn up to some date t at which he invests in the project. If this investment date is such that $Ae^{rt} \geq I$ (i.e., $t \geq \tilde{t}$), the project is financed with internal funds. However, if $t < \tilde{t}$, the entrepreneur needs outside funds, which are supplied, say, by a bank. We assume that the loan contract is proposed by the

only bad news can be learnt may be interpreted in the following way: the value created by the project R is well-known, but there may be essential impairments which "kill" the value of the project. For instance, the entrepreneur is perfectly aware of the performance of a drug, car, or software, but needs to run clinical tests, crash tests, or design an alpha/beta version in order to confidently reject the presence of side-effects, safety risks, or bugs.

[7] As discussed in Section 6, one could equivalently assume costly experimentation.

[8] For instance, one could write $\lambda = k\tilde{\lambda}$, where $\tilde{\lambda}$ is private information to the entrepreneur, and k is an observable parameter which varies across industries.

entrepreneur to the bank at the date at which he wants to invest.[9] The contact specifies that the entrepreneur invests all his available wealth Ae^{rt}, and that investment takes place immediately.[10] In case of failure, both the entrepreneur and the bank get 0, as we assume players to be protected by limited liability. In case of success, the bank recoups $R - R_e$ as reimbursement. Finally, we assume perfect competition among banks. A consequence is that the bank demands an instantaneous rate of return of rdt, which goes to 0, as the loan is immediately reimbursed in case of success.

Importantly, we assume that the bank knows how long the entrepreneur has been learning at any point in time.[11] However, whether a jump revealing a bad project is observed by all parties or privately observed by the entrepreneur is irrelevant here, as the entrepreneur never solicits funding if he knows the project to be bad. Indeed, the cash he must invest out of pocket would then be lost for sure, no matter the terms of the loan.

3 Complete information benchmark

Before we turn to the signaling problem raised by private information on λ, let us first examine how λ affects the entrepreneur's behavior in the benchmark case of perfect information. In this case, even if the bank does not observe the entrepreneur's beliefs on the project, it can perfectly back out these beliefs using (2). Since the bank makes zero expected profit, the entrepreneur obtains the full NPV of the project regardless of the date at which he solicits funding. Let $W^*(\lambda, t)$ denote the expected discounted payoff at date 0 of a entrepreneur with learning ability λ when he invests at date t (conditional on no bad news). We have:

$$\begin{aligned} W^*(\lambda, t) &= e^{-rt} s(\lambda, t) \left(p^*(\lambda, t) R - I\right) \\ &= e^{-rt} \left(p_0 R - I + (1 - s(\lambda, t)) I\right) \end{aligned}$$

This expression evidences the trade-off faced by the entrepreneur between discounting and learning: while waiting delays the realization of the payoff, it also allows to keep the option not to invest alive. The value of this option depends on $1 - s(\lambda, t)$, that is, the

[9] We therefore deliberately ignore the possibility of contracts signed at earlier dates, which would allow potential transfers to take place before investment. Halac *et al.* (2013) focus more specifically on the optimal way to implement such contracts. See also Section 6.5, in which we discuss the role of this assumption.

[10] We will later show that it is always optimal for the entrepreneur to invest all his wealth in the project.

[11] See Section 6.3 to see how one could relax this assumption.

probability of not investing, a probability which increases in t and λ : waiting allows to avoid sinking the outlay I in case bad news accrues, and better learning increases the value of this option.

The optimal investment date t^* reflects this tradeoff. It is such that:[12]

$$t^* = \underset{t \geq 0}{\operatorname{argmax}}\, W^*(\lambda, t) = \max\left(-\frac{1}{\lambda}\ln\frac{p_0 r(R-I)}{(1-p_0)(\lambda+r)I}, 0\right). \tag{3}$$

It is immediate to see that t^* is nonincreasing in R, p_0 and r, and nondecreasing in I. However, t^* may be non-monotonic in λ :

Proposition 1 *The impact of λ on the optimal investment date t^* depends on the prior NPV of the project:*

- *If $p_0R - I < 0$, $t^*(\lambda)$ is a decreasing function of λ;*
- *If $p_0R - I \geq 0$, there exist λ^* and λ^{**}, with $0 \leq \lambda^* < \lambda^{**}$, such that $t^*(\lambda) = 0$ for $\lambda \leq \lambda^*$, $t^*(\lambda)$ is increasing on $[\lambda^*, \lambda^{**}]$ and decreasing on $[\lambda^{**}, +\infty)$.*

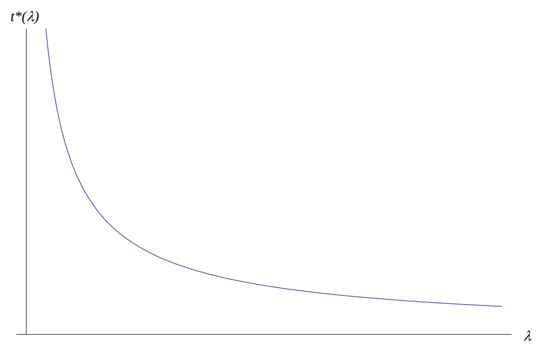

Figure 1: $t^*(\lambda)$ in the case $p_0R - I < 0$.

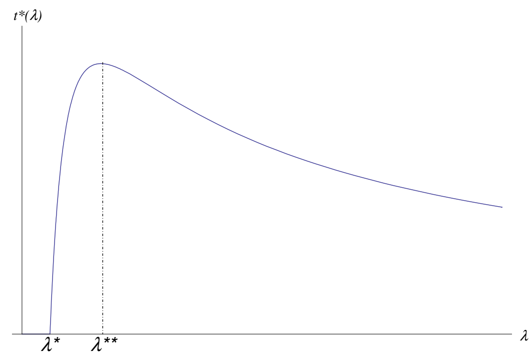

Figure 2: $t^*(\lambda)$ in the case $p_0R - I \geq 0$.

λ impacts the optimal investment date in two ways. An entrepreneur with a higher λ triggers investment when his beliefs reach a higher threshold, because the value of the option is higher.[13] Meanwhile, he also reaches a given threshold faster. When $p_0R - I < 0$, the latter effect always dominates, whereas both effects alternatively dominate when

[12]Notice that it is fine to solve this stopping problem by maximizing the date-0 expected payoff, since the entrepreneur perfectly forecasts at date 0 his conditional beliefs at all future dates.

[13]Notice that, with the same model, Décamps and Mariotti (2004) express the optimal strategy as a stopping rule: the entrepreneur triggers investment when the beliefs $p^*(\lambda, t)$ first hit a threshold $\hat{p}(\lambda) = \frac{(\lambda+r)I}{rR+\lambda I}$. It is immediate to see that $\hat{p}(\lambda)$ increases in λ.

$p_0R - I \geq 0$. Notice that Halac *et al.* (2013, 2014) derive the same non-monotonicity result in models with costly information acquisition.[14] This non-monotonicity makes the signaling content of early investment intrinsically ambiguous, as it could stem from an entrepreneur with a high learning ability who has learnt fast, or from a slow-learning entrepreneur whose option has little value. In the next section, we explore in detail how the entrepreneur can signal his learning speed through the timing of his investment.

4 Incomplete information

We now assume λ to be private information: $\lambda \in \left\{\underline{\lambda}, \overline{\lambda}\right\}$, with $0 \leq \underline{\lambda} < \overline{\lambda}$, and $\Pr(\lambda = \overline{\lambda}) = q_0$. To simplify notation, let us denote $\overline{t}^* = t^*(\overline{\lambda})$ and $\underline{t}^* = t^*(\underline{\lambda})$.

4.1 Payoffs

The interest rate paid by the entrepreneur reflects the bank's beliefs over the quality of the project. When asked for credit, the bank knows how long the entrepreneur has been waiting, but does not know the true beliefs of the entrepreneur because of private information on λ. Let q denote the probability that the bank assigns to the entrepreneur being of type $\overline{\lambda}$. Conditional on no signal between 0 and t, the bank's perceived probability of success reads

$$p\left(q,t\right) = \frac{p_0}{qs\left(\overline{\lambda},t\right) + (1-q)s\left(\underline{\lambda},t\right)}. \tag{4}$$

The bank lends $I - Ae^{rt}$ and demands a rate of return of 0, so the amount which the entrepreneur retains in case of success, $R_e(q,t)$, is given by the following zero profit condition:

$$p\left(q,t\right)\left(R - R_e\left(q,t\right)\right) = I - Ae^{rt}. \tag{5}$$

The interest rate on the loan then equals $\frac{R-R_e(q,t)-(I-Ae^{rt})}{I-Ae^{rt}} = \frac{1-p(q,t)}{p(q,t)}$.

Let $W\left(\lambda,q,t\right)$ denote the expected discounted payoff at date 0 of type $\lambda \in \{\underline{\lambda},\overline{\lambda}\}$ when he invests at date t, and is perceived as type $\overline{\lambda}$ with probability q. If $t \geq \widetilde{t}$, the entrepreneur does not need outside funds, so asymmetric information has no bite, and

$$W\left(\lambda,q,t\right) = W^*\left(\lambda,t\right) \text{ for all } q.$$

[14] In their case, though, the case where t^* is decreasing in λ never obtains. Indeed, they focus on the case where costly experimentation is valuable, which is the "counterpart" of our positive NPV case. In the opposite case where experimentation is too costly, there is no learning, so the investment date does not reflect learning ability.

If $t < \widetilde{t}$, we have

$$W(\lambda, q, t) = e^{-rt} s(\lambda, t) \left(p^*(\lambda, t) R_e(q, t) - Ae^{rt} \right) \tag{6}$$

Using (4) and (5), this reads

$$W(\lambda, q, t) = W^*(\lambda, t) + p_0 e^{-rt}(I - Ae^{rt}) \left[\frac{1}{p^*(\lambda, t)} - \frac{1}{p(q, t)} \right] \tag{7}$$

It is easy to see that $p^*(\underline{\lambda}, t) = p(0, t) \leq p(q, t) \leq p(1, t) = p^*(\overline{\lambda}, t)$. This implies that $W(\overline{\lambda}, 1, t) = W^*(\overline{\lambda}, t)$ and $W(\underline{\lambda}, 0, t) = W^*(\underline{\lambda}, t)$, i.e., the entrepreneur's expected payoff under asymmetric information is the same as under perfect information as long as the bank holds true beliefs on λ, a consequence of competition among financiers. Otherwise, the bad type benefits from being perceived as a good type with positive probability, as part of the investment $(I - Ae^{rt})$ is then financed at a cheaper cost than under perfect information (the interest rate is $\frac{1-p(q,t)}{p(q,t)} < \frac{1-p^*(\underline{\lambda},t)}{p^*(\underline{\lambda},t)}$). Conversely, the good type suffers from being perceived as a bad type with positive probability. Overall, the bad (resp. good) entrepreneur gets an expected payoff equal to his full-information payoff plus (resp. minus) an information rent reflecting the bank's (mis)perception that he is a high type with probability q.[15]

Notice also that (6) can be rewritten

$$W(\lambda, q, t) = e^{-rt} p_0 R_e(q, t) - A + (1 - s(\lambda, t)) A$$

For a fixed investment date t and a fixed repayment $R - R_e(q, t)$, both entrepreneur types get a payoff (viewed from date 0) of $e^{-rt} R_e(q, t)$ in case the project is of high quality. Therefore, the only difference between entrepreneurs with different learning abilities lies in the value of their options to wait: the good type (rightly) abstains from investing with probability $1 - s(\overline{\lambda}, t)$, in which case he saves the outlay (A in present value), while the bad type does so only with probability $1 - s(\underline{\lambda}, t)$. Let us define $f(t) \equiv s(\underline{\lambda}, t) - s(\overline{\lambda}, t) = (1 - p_0)(e^{-\underline{\lambda} t} - e^{-\overline{\lambda} t})$. We derive:

$$\forall t < \tilde{t}, \;\; W\left(\overline{\lambda}, q, t\right) - W(\underline{\lambda}, q, t) = Af(t) \tag{8}$$

[15] This can also be seen by remarking that $p_0(I - Ae^{rt}) \left[\frac{1}{p^*(\lambda,t)} - \frac{1}{p(q,t)} \right]$ can be rewritten $p_0 \left(R_e(q, t) - R_e(1, t) \right) < 0$ for $\overline{\lambda}$, and $p_0 \left(R_e(q, t) - R_e(0, t) \right) > 0$ for $\underline{\lambda}$.

One sees that f is nonnegative, as the good type invests in the project less often than the bad type. In addition, $f(0) = 0$, as both types differ only to the extent that some learning has taken place. Finally, it is easy to show that f is single-peaked and reaches a maximum at

$$t_0 \equiv \frac{\ln \overline{\lambda} - \ln \underline{\lambda}}{\overline{\lambda} - \underline{\lambda}} > 0.$$

The single-peakedness of f has to do with the fact that learning exhibits decreasing returns: the probability of learning bad news before t increases with t, but the marginal increase (the density) is decreasing with time. This implies that the comparative learning dynamics is characterized by two phases: a first phase in which the good type learns more (as time goes by, the good type becomes increasingly more optimistic than the bad type), and a second phase in which the bad type catches up: in the limit, there is perfect learning for any positive λ, so the difference between types becomes negligible.

Before we turn to the equilibrium analysis, let us notice that asymmetric information has no bite when $\overline{t}^* = 0$ and when $\overline{t}^* > \tilde{t}$: if $\overline{t}^* = 0$, then a fortiori $\underline{t}^* = 0$, using (3), so both types can reach their complete information payoff by investing at date 0, in which case the bank's beliefs are irrelevant ($p(q, 0) = p_0$ for all q); if $\overline{t}^* \geq \tilde{t}$, the good entrepreneur can self-finance the project, while securing his complete information payoff. In the following, we therefore restrict attention to $0 < \overline{t}^* < \tilde{t}$.

4.2 Equilibrium definition and concept

We look for perfect Bayesian equilibria satisfying D1. Whenever D1 is not enough to guarantee uniqueness, we select the Pareto-dominant equilibrium, or least-cost equilibrium.[16] A pure-strategy equilibrium features investment dates $\underline{t}$ and $\overline{t}$ (conditional on no news) for types $\underline{\lambda}$ and $\overline{\lambda}$, and a belief function $q(t)$, which assigns a probability that investment at date t comes from type $\overline{\lambda}$.[17]

[16] D1 imposes to attribute a deviation to some date t to the type with the stronger incentive to deviate to t. We will see that, whenever there is equilibrium multiplicity, the only equilibria are separating, meaning that they can be Pareto-ranked.

[17] Mixed strategies in such a continuous time game are not obvious to define, but we will later show that a mixed strategy equilibrium must involve one type randomizing between exactly two pure strategies, and the other type playing a pure strategy. See Lemma 4 in the Appendix.

4.3 Separating equilibria

We first look for separating equilibria where $\underline{t} \neq \bar{t}$. First of all, it is standard that, in any separating equilibrium, we must have $\underline{t} = \underline{t}^*$.[18]

A separating equilibrium $(\underline{t}^*, \bar{t})$ exists if and only if the following constraints hold:

$$\begin{cases} W(\overline{\lambda}, 1, \bar{t}) \geq W(\overline{\lambda}, 0, \underline{t}^*) & \text{(9a)} \\ W(\underline{\lambda}, 0, \underline{t}^*) \geq W(\underline{\lambda}, 1, \bar{t}) & \text{(9b)} \\ W(\overline{\lambda}, 1, \bar{t}) \geq W(\overline{\lambda}, q(t), t) \qquad \text{for all } t \notin \{\underline{t}^*, \bar{t}\} & \text{(9c)} \\ W(\underline{\lambda}, 0, \underline{t}^*) \geq W(\underline{\lambda}, q(t), t) \qquad \text{for all } t \notin \{\underline{t}^*, \bar{t}\} & \text{(9d)} \end{cases}$$

These incentive constraints specify that each type must prefer his equilibrium action to that of the other type, and has no profitable off path deviation.

The first thing to notice is that we must have $\bar{t} \leq \tilde{t}$. Otherwise, the good type could deviate to $\tilde{t}$: he could still self-finance the project, but would get a strictly higher payoff, since $\bar{t}^* < \tilde{t}$. This implies, taking (8) at $q = 1$:

$$W\left(\underline{\lambda}, 1, \bar{t}\right) = W^*\left(\overline{\lambda}, \bar{t}\right) - Af(\bar{t}). \tag{10}$$

$Af(t)$ measures the difference in option values, hence captures the cost for the slow-learning type to mimic the fast-learning type. Since the least cost separating equilibrium is the one which maximizes the expected payoff of the good type, the first avenue is to check whether the complete information dates $(\underline{t}^*, \bar{t}^*)$ can be equilibrium strategies. We establish the following result:

Proposition 2 $(\underline{t}^*, \bar{t}^*)$ *is an equilibrium if and only if* $A \geq A_0 \equiv \frac{W^*\left(\overline{\lambda}, \bar{t}^*\right) - W^*\left(\underline{\lambda}, \underline{t}^*\right)}{f(\bar{t}^*)}$

The entrepreneur can therefore reach his complete information payoff whenever the cash constraint is sufficiently soft. When a high share of the investment is internally financed (A increases), the benefit of cheap credit does not compensate the loss due to the inefficiency of the investment policy, and mimicking is not a concern. Notice that $A_0 \in (0, I)$, so the complete information payoffs are always attainable for A large enough, and never attainable when A is too small.

[18] If $\underline{t} \neq \underline{t}^*$, type $\underline{\lambda}$ could always increase his payoff by playing $\underline{t}^*$: even it is does not improve the bank's beliefs, it yields a higher expected payoff.

Before we turn to the analysis of other separating equilibria, let us remark that D1 allows us to further restrict the set of possible equilibrium dates $\bar{t}$, as Lemma 1 shows.

Lemma 1 *In any separating equilibrium, the good type invests at a date comprised between* $\bar{t}^*$ *and* t_0 : $\bar{t} \in [\min(t_0, \bar{t}^*), \max(t_0, \bar{t}^*)]$.

Given that f measures how much the good type prefers investment at a given date as compared to the bad type, and because f is single-peaked in t_0, a deviation to an off-equilibrium date between t_0 and $\bar{t}$ ("in the direction of t_0") is relatively more beneficial to the good type (for any out-of-equilibrium associated to this deviation). Therefore, D1 imposes to attribute such a deviation to the good type. Given this restriction, it is easy to find profitable deviations if $\bar{t} \notin [\min(t_0, \bar{t}^*), \max(t_0, \bar{t}^*)]$: by moving in the direction of t_0, the good type could secure a higher expected payoff, while still being perceived as good.

Let us define

$$\begin{aligned} \Lambda_H &= \left\{ \left(\underline{\lambda}, \overline{\lambda}\right) \in \mathbb{R}_+^2 | \overline{\lambda} > \underline{\lambda} \text{ and } \bar{t}^* > t_0 \right\} \\ \Lambda_D &= \left\{ \left(\underline{\lambda}, \overline{\lambda}\right) \in \mathbb{R}_+^2 | \overline{\lambda} > \underline{\lambda} \text{ and } \bar{t}^* \leq t_0 \right\}. \end{aligned}$$

Notice that, since neither t_0 nor $\bar{t}^*$ depend on A, the regions Λ_H and Λ_D are independent of A. Therefore, a direct implication of Lemma 1 is:

Corollary 1 *Suppose* $A < A_0$. *If* $(\underline{\lambda}, \overline{\lambda}) \in \Lambda_H$ *(resp.* Λ_D*), any separating equilibrium must involve hurried (resp. delayed) investment:* $\bar{t} \in [t_0, \bar{t}^*)$ *(resp.* $\bar{t} \in (\bar{t}^*, t_0]$*).*

The investment date must reflect the compromise between the preferred investment strategy of the good type (invest at $\bar{t}^*$) and the call for incentives (which requires distorting investment in the direction of t_0 to increase the difference in option values). As a consequence, separation may involve investing earlier or later than $\bar{t}^*$ according to where $\bar{t}^*$ lies as compared to t_0. Let us now characterize how the ordering of t_0 and $\bar{t}^*$ relates to the primitives of the model.

Lemma 2 Λ_H *and* Λ_D *are characterized as follows:*

- *If* $\overline{\lambda} \leq \frac{ep_0 r(R-I)}{(1-p_0)I} - r$, $(\underline{\lambda}, \overline{\lambda}) \in \Lambda_D$ *for all* $(\underline{\lambda}, \overline{\lambda})$;
- *If* $\overline{\lambda} > \frac{ep_0 r(R-I)}{(1-p_0)I} - r$, *there exists a unique* $\underline{\lambda}_0$ *such that* $(\underline{\lambda}, \overline{\lambda}) \in \Lambda_D$ *if and only if* $\underline{\lambda} \leq \underline{\lambda}_0$, *where* $\underline{\lambda}_0$ *is a decreasing function of* $\overline{\lambda}$.

Figure 3 depicts how the set of types is partitioned into the two regions Λ_H and Λ_D.

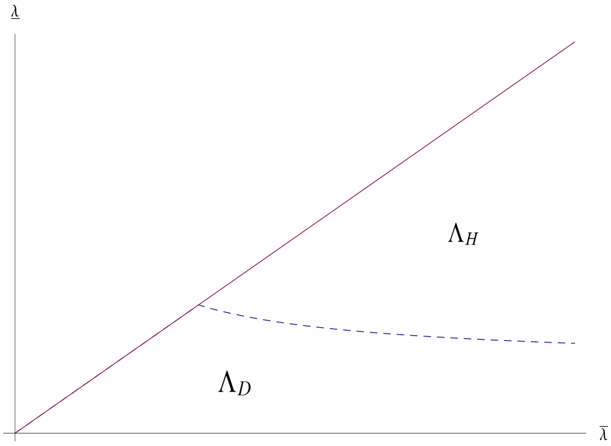

Figure 3: The regions Λ_H and Λ_D.

To understand the intuition behind Lemma 2, recall that f reflects the comparative learning dynamics: in a first phase, beliefs of both types diverge apart, with the good type learning faster than the bad type; in a second phase, the bad type catches up on beliefs.[19] In region Λ_H, both types learn sufficiently fast. This means that the phase during which the good type learns faster stops early. In this case, signaling learning ability imposes to invest early to make sure that the bad type is sufficiently less confident than the good type about the project. Conversely, in Region Λ_D, the slow-learning type learns little enough, so the first phase stops later, and the good type should exploit as much as possible his comparative learning advantage by waiting longer.

While the necessary conditions given by Lemmas 1 and 2 are helpful to better understand how the incentive-compatibility constraints shape equilibrium behavior, we still need to derive necessary and sufficient conditions for a separating equilibrium, which we do now:

Proposition 3 *There exists a separating equilibrium if and only if* $A \geq A_1 \equiv \max\left(0, \frac{W^*(\overline{\lambda}, t_0) - W^*(\underline{\lambda}, \underline{t}^*)}{f(t_0)}\right)$

In a separating equilibrium, the sorting of types is achieved by making sure that the difference between each type's beliefs on the project is large enough that it is worthwhile for the good type to invest his cash A out of pocket, but too costly for the bad type. This is why a separating equilibrium is only possible when the entrepreneur has enough cash.[20]

[19] Formally, $p^*(\overline{\lambda}, t) - p^*(\underline{\lambda}, t)$ is increasing and then decreasing.

[20] Notice though that one may have $A_1 = 0$, in which case there exists a separating equilibrium even if the entrepreneur has no cash. See Section 4.6 for a discussion on this.

It is easy to see that $A_1 \leq A_0$. We immediately derive from Corollary 1 and Proposition 3:

Corollary 2 *If $A_1 \leq A < A_0$, a separating equilibrium exists, and involves hurried (resp. delayed) investment if $(\underline{\lambda}, \overline{\lambda}) \in \Lambda_H$ (resp. Λ_D).*

While the existence of a separating equilibrium depends on the severity of the information problem measured by A, the direction of the distortion depends on the learning environment, i.e., on whether $(\underline{\lambda}, \overline{\lambda})$ belongs to Λ_H or Λ_D, but is independent of A.

4.4 Pooling and semi-pooling equilibria

We now characterize pooling and semi-pooling equilibria:

Proposition 4 *If $A \geq A_1$, there cannot be pooling in equilibrium (any equilibrium is separating).*

In addition, $\exists\, A_2 \leq A_1$ such that:

- *If $A_2 \leq A < A_1$, the unique equilibrium is such that $\overline{t} = t_0$, and $\underline{\lambda}$ randomizes between t_0 and $\underline{t}^*$,*
- *If $A < A_2$, the unique equilibrium is pooling: $\underline{t} = \overline{t} = t_0$.*

In the Appendix, we derive the cutoff value A_2 :

$$A_2 = \max\left(0, Ie^{-rt_0} - \frac{W^*(\underline{\lambda}, \underline{t}^*) - W^*(\underline{\lambda}, t_0)}{q_0 f(t_0)}\right).$$

Notice that the pooling equilibrium never exists if $A_2 = 0$, i.e., when $q_0 \leq \frac{W^*(\underline{\lambda}, \underline{t}^*) - W^*(\underline{\lambda}, t_0)}{Ie^{-rt_0} f(t_0)}$. This corresponds to instances where the prior probability q_0 is small enough, in which case the bad type is unwilling to distort his investment strategy, as the gain of being perceived as the average type is too small.

Two findings emerge from Proposition 4. First, the equilibrium cannot involve any pooling when a separating equilibrium exists. Second, when there is pooling in equilibrium, it must be at date t_0. This springs from Lemma 1, which imposes the distortion to be "capped" at t_0. We derive the following Corollary:

Corollary 3 *The equilibrium is unique*

The only possible range of multiplicity is $A \geq A_1$. In this case, there may be a continuum of separating equilibria, but Proposition 4 shows that pooling is then impossible. Therefore, all these equilibria can be Pareto-ranked, and there is consequently a unique least-cost separating equilibrium.

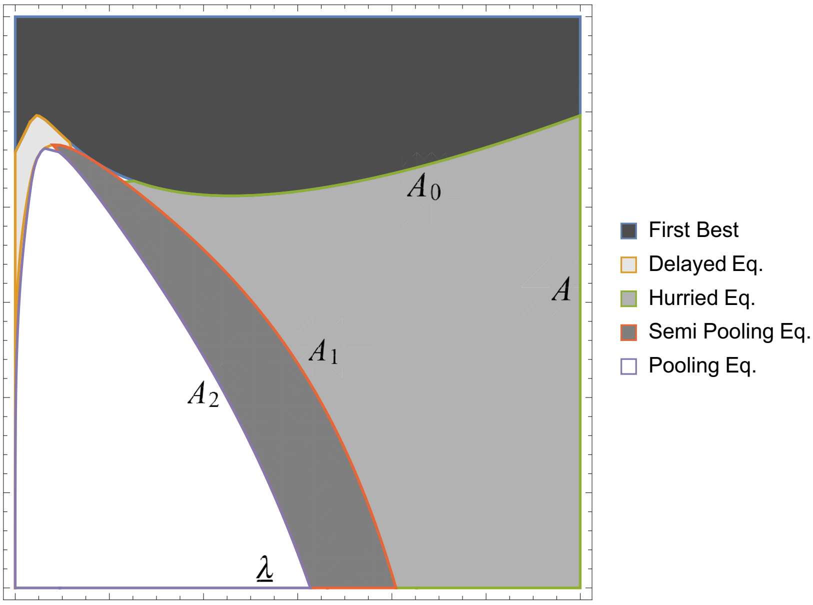

Figure 4: Equilibrium characterization

In Figure 4, we represent the different equilibrium regions in the space $(\underline{\lambda}, A)$. The figure displays that higher values of A make it more likely that the first best is an equilibrium, and, if not, that there exists a separating equilibrium. It also shows that separation is obtained by delaying investment when $\underline{\lambda}$ is sufficiently small, and by hurrying investment otherwise, regardless of the value of A.

4.5 The role of the cash constraint

The equilibrium strategies reflect the severity of the incentive problem, which is measured by A. When the cash constraint is soft, the complete information payoffs are attained. Otherwise, the good type needs to distort his investment date in the direction of t_0 in order to prevent mimicking by the bad type. When the information problem becomes too severe (A is too small), the only solution for the good type is to invest at t_0 to maximize the difference in option values. When this is not sufficient to achieve separation, the good type must be (fully or partially) pooled with the bad type. Proposition 5 establishes formally that the distortion incurred by the good type increases with the cash shortage.

Proposition 5 *If $t_0 < \bar{t}^*$ (resp. $\bar{t}^* < t_0$), $\bar{t}$ is nondecreasing (resp. nonincreasing) in A :*

- *For $A < A_1$, $\bar{t} = t_0$*
- *For $A \in [A_1, A_0]$, $\bar{t}$ is increasing (resp. decreasing) in A*
- *For $A > A_0$, $\bar{t} = \bar{t}^*$*

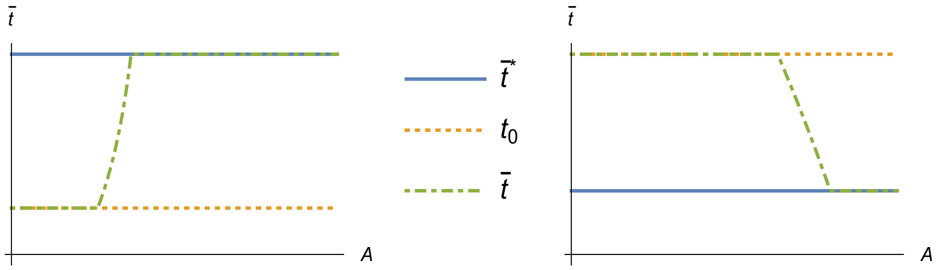

Figure 5: The impact of the cash constraint

One sees from $W(\underline{\lambda}, 1, t) = W^*(\overline{\lambda}, t) - Af(t)$ that, whenever A increases, changes in t have a larger impact on the incentive constraint of the bad type, which reduces the distortion. A captures the stake of the entrepreneur in his project, i.e., the share of the investment which is financed internally. As the cash shortage problem improves (that is, as A increases), the benefit from fooling the financier decreases, and the cost from distorting the timing away from the preferred timing policy increases. By improving the sorting of types, a higher A therefore attenuates the information problem, and decreases the welfare loss. Since Proposition 5 only concerns the good type, let us now derive how A affects total expected welfare:

Corollary 4 *The good entrepreneur's expected payoff $W(\overline{\lambda}, q(\bar{t}), \bar{t})$ and the expected total welfare $q_0 W(\overline{\lambda}, q(\bar{t}), \bar{t}) + (1 - q_0) W(\underline{\lambda}, q(\underline{t}), \underline{t})$ are nondecreasing in A*

Figure 6 illustrates this result. Pooling hurts the good type, but benefits the bad type: although the bad type must then also suffer some distortion, by investing at date t_0 with positive probability, he benefits from being pooled with the good type.

Notice from this result that it is clear that the entrepreneur should invest all his wealth in the project. Indeed, there is no point for the good type to borrow more than needed (i.e., to borrow $I - \tilde{A}e^{rt}$ at date t, with $\tilde{A} < A$), as this would just make mimicking more tempting, and result in a higher distortion. Since it is a (weakly) dominant strategy for

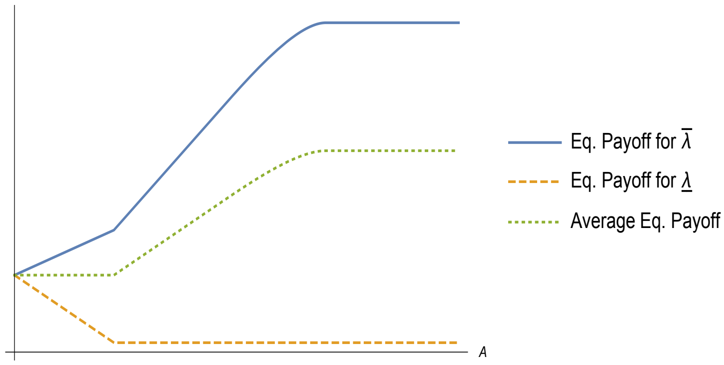

Figure 6: Equilibrium payoffs

the good type to invest all his cash, the bad type could never reveal positive information by borrowing more than needed.[21] Therefore, we would derive the same results if the entrepreneur could choose both his investment date and how much to borrow.[22]

4.6 Relationship with the complete information benchmark

4.6.1 Distortions

Proposition 1 establishes that there may be a non-monotonicity in the investment dates under complete information: a fast-learning entrepreneur possibly invests earlier or later than a slow-learning one. Intuitively, the ranking of the first best investment dates should affect the way separation is achieved under asymmetric information. When $\overline{t}^* < \underline{t}^*$, investing earlier than $\overline{t}^*$ should be a way for the good type to increase the cost of mimicking for the bad type. And conversely if $\underline{t}^* < \overline{t}^*$. However, the difference between each type's preferences under perfect information is not fully captured by the ranking of the first-best investment dates, but instead by the (non-monotone) function $e^{-rt}f(t)$. Indeed, we have:

$$W^*(\overline{\lambda}, t) - W^*(\underline{\lambda}, t) = Ie^{-rt}f(t) \text{ for all } t. \tag{11}$$

Remarking that the incentive for the bad type to mimic the good type is measured by $Af(t)$, there is only a partial relationship between the complete information ordering

[21]However, the fact that the bad entrepreneur's expected payoff is decreasing in A raises the concern that he could squander his cash to increase his equilibrium payoff in the signaling game. This would be a concern only if A were not observable; but, in this case, entrepreneurs with little cash should rationally be suspected of being bad types.

[22]The fact that both types invest all their cash (that is, pool on this dimension) contrasts with what would happen in a screening model, where the principal could precisely prescribe to invest all his cash only to the good type, so as to elicit information on the agent's confidence in the project. See also Section 6.5 on this issue.

and the direction of the distortion under asymmetric information: whenever $\underline{t}^* < \overline{t}^*$, a separating equilibrium must involve delayed investment by the good type.[23] This is in line with the intuition given by the ranking of the first best investment dates. However, in the other case where $\overline{t}^* \leq \underline{t}^*$, the equilibrium could possibly involve either hurried or delayed investment. The preference of each type over timing decisions is then a sorting force *per se*, as it is intrinsically costly for an entrepreneur of a given type to invest at the preferred date of the other type. Accordingly, the preferences may be sufficiently different for a separating equilibrium to exist even when the entrepreneur has no cash (this is the case when $A_1 = 0$).[24] The fact that the entrepreneur invests his own money into the project is therefore a complementary sorting force, which helps achieve separation, but is not always necessary.

4.6.2 Reversals

The distortions created by asymmetric information may result in two possible types of reversal as compared to the complete information case:[25]

- A reversal of the ranking of investment dates: this happens when the good type invests later than the bad type under asymmetric information, while he would invest earlier under perfect information: $\overline{t}^* < \underline{t}^* < \overline{t}$.[26]
- A reversal of the ranking of beliefs: this happens when the good type is less optimistic on the project upon investing than the bad type: $p^*(\overline{\lambda}, \overline{t}) < p^*(\underline{\lambda}, \underline{t}^*)$, as shown in Figure 7.[27]

4.6.3 The impact of $\overline{\lambda}$ on investment

In order to compare the shape of the investment dates under complete information (given by Figures 1 and 2) and under incomplete information, let us derive how $\overline{t}$ varies when $\overline{\lambda}$ changes, holding $\underline{\lambda}$ fixed. In case the first best is an equilibrium, $\overline{t} = \overline{t}^*$ may be decreasing

[23] One easily sees that $\underline{t}^* < \overline{t}^* \Rightarrow \overline{t}^* < t_0$, using (11) and the fact that $e^{-rt}f(t)$ reaches a maximum at a point below t_0.

[24] This does not mean that the cash invested by the entrepreneur does not play a sorting role when $A_1 = 0$. Indeed, the distortion is lower when A increases, as we have seen in the previous section.

[25] See the proofs in Section 8.10 of the Appendix for more details on the conditions for such reversals.

[26] Notice that this result hints that the monotonicity condition typical of a screening problem would be violated. See notably Halac *et al.* (2013) on this issue.

[27] Under complete information, $p^*(\lambda, t^*(\lambda))$ is increasing in λ, so an entrepreneur with a higher option value is always more confident upon investing.

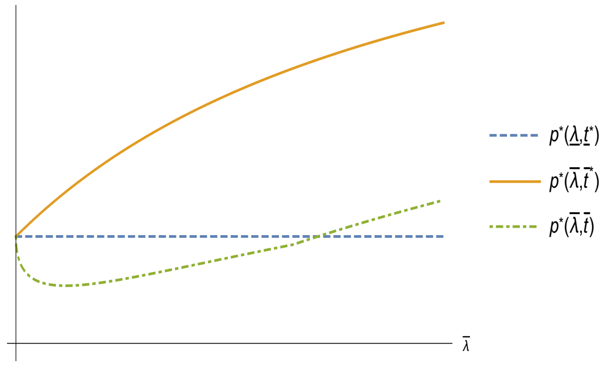

Figure 7: Reversal of belief monotonocity

or single-peaked in $\overline{\lambda}$ (see Proposition 1); in case the equilibrium is pooling or semi-pooling, the investment date $\overline{t} = t_0$ is decreasing in $\overline{\lambda}$. In the last case where a separating equilibrium with distortion exists, we derive the following result:

Proposition 6 *In a separating equilibrium involving delayed (resp. hurried) investment, the investment date $\overline{t}$ is increasing (resp. decreasing) in $\overline{\lambda}$.*

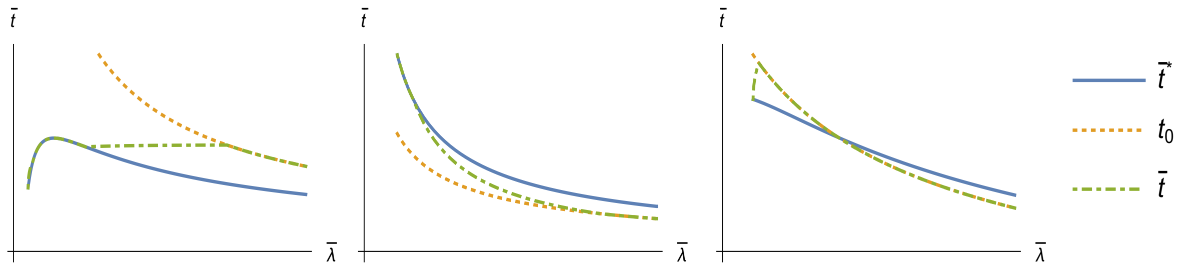

Figure 8: The equilibrium investment date $\overline{t}$ as a function of $\overline{\lambda}$

Figure 8 illustrates the results of this Section. In the left figure, we depict situations where $\underline{t}^* \leq \overline{t}^*$, in which case a separating equilibrium must involve delayed investment, and $\overline{t}$ is then increasing in $\overline{\lambda}$ over the zone where such a separating equilibrium exists. This implies in particular that the investment date may change monotonicities twice as $\overline{\lambda}$ increases. In the other two figures, we depict situations where $\overline{t}^* < \underline{t}^*$ in which case a separating equilibrium may involve either hurried or delayed investment. An important implication of Proposition 6 is that, whenever $\overline{t}^*$ is decreasing in $\overline{\lambda}$ and $\overline{t}$ is increasing in $\overline{\lambda}$, the welfare loss, $q_0\left(W^*(\overline{\lambda},\overline{t}^*) - W^*(\overline{\lambda},\overline{t})\right)$, must be increasing in $\overline{\lambda}$, meaning that improvements in the learning technology of the (good) entrepreneur result in more severe inefficiencies.

4.7 The role of the prior NPV

In this section, we examine the impact of the variables governing the NPV of the project, i.e., p_0, R and I. In order to underline the role played by the date-0 NPV, and to simplify the exposition, we focus attention on the case $\underline{\lambda} = 0$. With this specification, $f(t) = (1-p_0)(1-e^{-\overline{\lambda}t})$ is increasing in t, so we have $t_0 = \infty$. This implies $A_1 = 0$, meaning that the equilibrium is always separating. In addition, the equilibrium never involves hurried investment.

Proposition 7 *Consider* $\underline{\lambda} = 0$.

- *If* $A \geq I(\frac{r}{\overline{\lambda}+r})^{r/\overline{\lambda}}$, $(\underline{t}^*, \overline{t}^*)$ *is an equilibrium for any* p_0;
- *If* $A < I(\frac{r}{\overline{\lambda}+r})^{r/\overline{\lambda}}$, *there exist* $(\underline{p}, \overline{p}) \in (0,1)^2$ *with* $\underline{p} < \frac{I}{R} < \overline{p}$ *such that the* $(\underline{t}^*, \overline{t}^*)$ *is an equilibrium if and only if* $p_0 \notin [\underline{p}, \overline{p}]$:
  - *If* $\underline{p} \leq p_0 < \frac{I}{R}$, *the (delayed) equilibrium investment date* $\overline{t}$ *is increasing in* p_0 *and* R, *and decreasing in* I;
  - *If* $\frac{I}{R} < p_0 < \overline{p}$, *the (delayed) equilibrium investment date* $\overline{t}$ *is decreasing in* p_0 *and* R, *and increasing in* I.

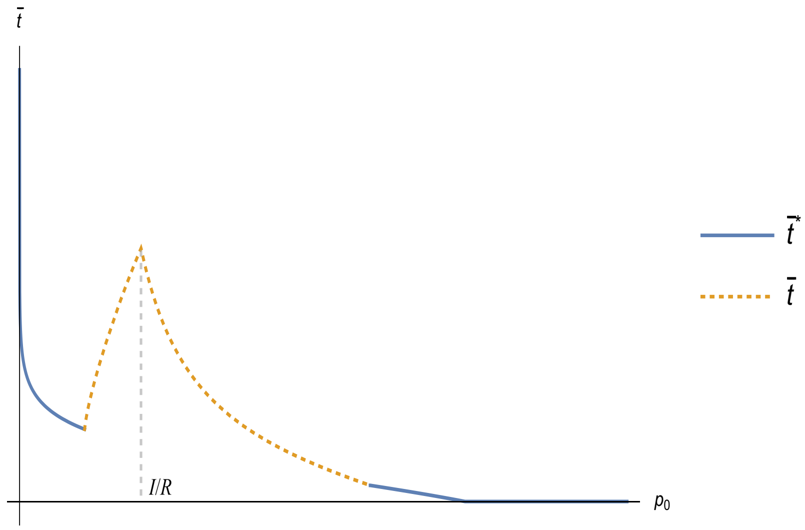

Figure 9: The equilibrium investment date $\overline{t}$ as a function of p_0

The equilibrium investment date is increasing in the NPV in the negative NPV range, and decreasing in the positive NPV range, as illustrated in Figure 9.[28] An increase in

[28]Notice though that this non-monotonicity is not an artifact of choosing $\underline{\lambda} = 0$. Indeed, we also derive it under some conditions in the general case, but the presentation of the results is simpler in this specific configuration. The results on the comparative statics with respect to p_0 in the general case are available upon request.

the value of the project may therefore result in a decrease in investment. This is because the equilibrium payoff of the bad type does not change with the NPV as long as it is negative, but the expected payoff of the good type increases with the NPV, which increases incentives to mimic, hence the higher distortion. Another implication of Proposition 7 is that the welfare loss due to the financial friction, $q_0\left(W^*(\overline{\lambda}, \overline{t}^*) - W^*(\overline{\lambda}, \overline{t})\right)$, is increasing in the NPV over the range where $\overline{t}$ is increasing. This raises the concern that markets where potential innovations have higher value are plagued by larger inefficiencies.

5 Empirical predictions

The first prediction of the model is that the cash constraint does affect investment. Firms with similar projects but different levels of cash do not necessarily invest the same. This is consistent with the finding that cash-rich firms have an investment policy which is less sensitive to their net worth than cash-constrained firms (see Hubbard (1998) for a survey). In addition, our model predicts more specifically that inefficient investment may take both the form of over-investment or under-investment according to the shape of the learning curves in the market. In industries with slow learning, cash-constrained firms should invest later (or less) than unconstrained firms, and succeed with a higher probability. This is consistent with Guedj and Scharfstein (2004), who show that small drug companies with less initial cash are less likely to move to subsequent phases of clinical tests, and are more likely to succeed in these phases, that is, cash-constrained firms experiment longer. This is also consistent with the fact that long deferments of patent examination are particularly frequent in the pharmaceutical and chemical industries (Henkel and Jell, 2010).[29] In industries characterized by fast learning (i.e., where even slow learners learn fast enough), cash-constrained firms should invest earlier than unconstrained firms, and have a lower probability of success. In terms of the timing of patenting decisions, this implies that firms in industries where learning is relatively fast (e.g. software) should be more prone to soliciting accelerated patent examination procedures. Unfortunately, there has been little evidence on the characteristics of firms which file for accelerated procedures so far.[30]

[29]Some patent offices allow patent applicants to solicit accelerated and/or deferred examination of their application, lowering or expanding *de facto* the duration of the experimentation period. Patents are often perceived as a way for cash-poor firms to secure financing by signaling their quality (Hall and Harhoff, 2012).

[30]Harhoff and Stoll (2014) exploit as a natural experiment the fact that the European Patent Office has switched from a regime where accelerated examination procedures were publicly disclosed to a regime

Overall, our model suggests that an empirical analysis on the impact of cash constraints on investment should group firms according both to how financially constrained they are, and to the learning speed in the industry. Indeed, our results hint that an analysis where firms are grouped according to their net worth only would possibly underestimate the impact of the cash constraint, by pooling together firms which underinvest and firms which overinvest.

Finally, our model also predicts that the value of the project may have a counterintuitive impact on investment: a higher expected value may indeed sometimes decrease investment. Relatedly, we evidence two distinct "ranges of inaction" in which investment does not respond to a change in the expected profitability of the project: one range corresponds to the situation in which the option to wait is not exerted (if investment takes place at date 0, an increment in the NPV does not change the entrepreneur's behavior, both under complete or incomplete information); the other range corresponds to the zone where there is (full or partial) pooling (the entrepreneur is fully constrained by incentives, and keeps investing at date t_0, even when the NPV marginally increases). Therefore, the non-responsiveness of investment to changes in the value of the investment may have two very different causes: the option-like feature of investment and the cash constraint. In firms suffering from cash constraints, the non-responsiveness is more likely to originate from an incentive problem, while in firms holding projects which are easier to revert (for instance, when there is a liquid second-hand market for assets), the option has a smaller value, and the non-responsiveness should rather reflect the desire of the entrepreneur to reap the benefits from investment as soon as possible. It would be interesting to test these predictions empirically.

6 Discussion

6.1 Rate of capitalization of the entrepreneur's cash

One may wonder whether our results would hold if we instead assumed that the cash of the entrepreneur A is capitalized at some different rate $r_0 \leq r$. In this case, the function f would become $(1-p_0)\, e^{-(r-r_0)t} \left(e^{-\underline{\lambda} t} - e^{-\overline{\lambda} t}\right)$, which is still a single peaked function of t, with a maximum reached at a $\tilde{t}_0 \leq t_0$, where $\tilde{t}_0$ increases in r_0. Therefore, our results would go through: the region where hurried investment obtains would simply expand, and the

where they are kept secret. But they do not focus on differences in the pool of applicants across industries.

region characterized by delayed investment shrink. This highlights how the capitalization of the entrepreneur's cash affects the incentive problem. Indeed, waiting longer is more of an effective signaling strategy when r_0 is higher, since the amount that is internally financed increases every thing else equal, hence the cost incurred by the bad type when mimicking the good type.[31]

6.2 Costly experimentation

It is also easy to show that our results would hold with costly experimentation. For instance, we derive an essentially similar result when there is no discounting and learning involves a flow cost c.[32] In this case, since an entrepreneur who invests later has to pay higher experimentation costs, the cash he has left to finance the project shrinks with time.[33] This effect mirrors the effect of cash capitalization described in the previous paragraph: here, the cash of the entrepreneur depreciates, which makes the hurried investment equilibrium more likely everything else equal.

6.3 Non-observability of date 0

Since the investment date is the signal which the entrepreneur uses to display his confidence, it is critical that the bank is able to observe how long exactly the entrepreneur has been waiting, i.e., "knows date 0". Suppose instead that the bank does not know the exact waiting time, but that an entrepreneur who has been waiting for a length T could provide hard evidence that he has been waiting at least for any length $\tilde{T} \leq T$.[34] In other words, the entrepreneur could possibly understate, but not overstate his waiting time. In this case, our equilibrium with hurried investment would collapse: if the bank believes an entrepreneur who pretends that he has been waiting for $\bar{t}$ to be a high type, then the entrepreneur should wait until $\bar{t}^*$, but pretend he has only been learning for $\bar{t}$. However, an equilibrium with delayed investment would be robust to this altered information structure, since such gaming is unprofitable in this direction. Therefore, the non-observability of date 0 would create some asymmetry between the (robust) delayed equilibrium and the

[31]Notice also that a separating equilibrium becomes more likely when r_0 increases, as the entrepreneur has more cash to invest everything else equal.

[32]The proof is omitted for concision, but is available upon request.

[33]This holds of course only as long as one assumes that the experimentation costs are monetary.

[34]For instance, the entrepreneur could exhibit past trial outcomes, which do not provide information on the quality of the project, but evidence that the entrepreneur was already learning on the project at the time of the trial.

(non robust) hurried equilibrium.

6.4 Good news model

Our assumption that "no news is good news" makes the analysis simpler, as there is a one-to-one relationship between dates and beliefs as long as the entrepreneur has not received bad news. Let us try to conjecture what would happen in a "good news" model, i.e., in a learning environment where the Poisson process has intensity λ in case the project is of high quality, and 0 otherwise. Under complete information, it is easy to see that the first best rule consists of investing either at date 0 or only upon learning good news. We then obtain an equivalent of our non-monotonicity result: if the NPV is negative, investment occurs at the first date at which good news arises. This date is a random variable, but it comes earlier in expectation when λ increases. If the NPV is nonnegative, there is immediate investment if λ is small enough, while the expected investment date decreases in the range where the option is exerted (λ is large enough). In the presence of a cash constraint, asymmetric information is irrelevant as long as jumps in the Poisson process are observable to the bank. Indeed, the entrepreneur solicits funding only in instances where types with different learning speeds have the same beliefs on the project, that is, either at date 0, or upon observing a signal perfectly revealing a good project. However, if jumps in the Poisson process are unobservable to the bank, it is impossible to back out beliefs from the investment date, since the first date at which there is a jump is stochastic. This is true even with no uncertainty on the entrepreneur's type, so that private learning gives rise to an interesting signaling problem even under perfect information on λ. We conjecture that the bank should not grant funding before some critical date, in order to make sure that the entrepreneur is sufficiently pessimistic that he does not invest his own cash if he has not learnt good news yet. In case there is additional private information on λ, this also guarantees that the bad type still prefers to invest at date 0 (when this is his preferred strategy under complete information) than waiting to secure cheaper credit. In any case, we conjecture that asymmetric information would generate inefficient delaying of investment.

6.5 Commitment

By focussing on loan contracts signed at the date t at which the investment is made, we make important restrictions on commitment, communication, and instruments: first, we implicitly rule out commitment power, which would allow the entrepreneur and the financier to agree *ex ante* on some contractual terms; second, we ignore the possibility for the entrepreneur to reveal information on his type using other messages than the timing decision; third, we rule out possible transfers prior to the investment date. In order to assess how sensitive our results are to these various restrictions, we might alternatively consider a screening problem, in which a monopolistic bank proposes a contract at date 0 to the entrepreneur. First of all, in order to make things comparable with our model (that is, to "test" the role of our assumption on commitment), one could impose that the entrepreneur always invests all his cash A, and that the only contractible variables are the payoff of the entrepreneur in case of success, and the investment date. This framework is the natural screening counterpart to our signaling model. In such a context, the function f, which measures the difference in option values, captures the rent that the good type should be given not to mimic the bad type. In order to lower this rent, the investment timing of the bad type should be distorted away from t_0.[35] In addition, notice that the result of no distortion at the top does not always hold, as the incentive constraints imply a "monotonicity condition" $f(\bar{t}) \geq f(\underline{t})$, which sometimes imposes that the good type's investment date also be distorted in the direction of t_0, as in our signaling model. The distortion therefore reflects the incentive problem captured by the same function f, suggesting that our results are no artifact of the no-commitment assumption. Notice, however, that allowing for a wider message space or set of instruments would relax the incentive problem.[36] For instance, the bank could use how much the entrepreneur internally finances to screen the entrepreneur's information, relying on the higher willingness to invest his cash out of pocket of a more confident entrepreneur.

[35] A standard difference between screening and signaling two-type models, is that the type who suffers a distortion is the bad type in a screening model (no distortion at the top), and the good type in a signaling game. By the same logic, the distortions go in opposite directions in each model.

[36] For instance, Halac *et al.* (2013) allow for transfers at all dates prior to investment, and show that there are timing distortions only under the conjunction of moral hazard and adverse selection. In our model with adverse selection only, we would obtain the first best with such a richer set of instruments.

6.6 Other corporate finance applications

In our specification, the entrepreneur's objective is to minimize his cost of capital, which is a weighted average of the cost of internal funds given by the inverse of his true beliefs, and the cost of outside capital given by the inverse of the bank's beliefs over the project. It is easy to show that a slightly different version of the model, where the entrepreneur maximizes some average of the true expected value of the project (given his own private information) and the expected value of the project, such as perceived by the uninformed financier, would give analogous results. In other words, the problem we study is qualitatively similar to a problem of managerial myopia, in which the manager cares about both the true value of the firm and the stock price, or to a problem of optimal IPO timing, where the firm's owner (or the venture capitalist) chooses the date at which the firm goes public, so as to maximize some average of the value of current equity and the future IPO price. Our model is therefore suited to study the impact of information frictions on timing decisions in a more general class of corporate governance problems. This being said, our results contrast with related results established in this literature. Grenadier and Malenko (2011) and Bustamante (2012) both analyze models of signaling through investment timing, the former with an application to managerial myopia, the latter to IPO timing. They both find that asymmetric information generates hurried investment as compared to the first best, while we stress that both hurried or delayed investment may arise, depending on the shape of the learning curves in the market.

7 Conclusion

We consider a model in which a cash-constrained entrepreneur learns about a project, but at a speed which is his private information. The signaling problem arising from the conjunction of the information friction (private learning ability) and the financial friction (limited cash) results in the entrepreneur distorting his investment policy when the cash shortage is too severe. This distortion takes the form of hurried investment (under-experimentation) in markets with fast learning, and of delayed investment (over-experimentation) in markets where learning is slower.

The fact that both delayed and hurried investment may arise has to do with two noteworthy properties of our signaling game. First, the entrepreneur's decision endogenously affects the amount of relevant asymmetric information: the relevant asymmetric information is not the entrepreneur's type λ *per se*, but his beliefs about the project, which depend on both his (privately observed) type λ and the (observable) timing decision t. This relevant asymmetric information, measured by the difference in each type's beliefs, increases and then decreases with time.[37] This non-monotonicity explains why signaling may possibly involve hurried or delayed investment. Second, a distinctive feature of our modeling of private information on learning ability is that it is impossible to rank types according to their preference over investment dates: depending on their relative learning skills, a fast-learning type may be more willing or less willing than a slow-learning type to invest later, that is, the single-crossing property does not hold. While the analysis is made somewhat more complex, it also highlights in an intuitive way how signaling could be achieved by late or early investment, by relating the direction of the timing distortion to the speed of learning in the market. Both from the methodological point of view and in terms of predictions, our paper therefore offers a substantial contribution to the literature on learning/experimentation under asymmetric information.

References

Bebchuk, L. A. and Stole, L. A. (1993). Do short-term objectives lead to under- or overinvestment in long-term projects? *The Journal of Finance*, **48** (2), 719–730.

Bergemann, D. and Hege, U. (1998). Venture capital financing, moral hazard, and learning. *Journal of Banking & Finance*, **22** (6), 703–735.

— and — (2005). The financing of innovation: Learning and stopping. *RAND Journal of Economics*, **36** (4), 719–752.

Bouvard, M. (2014). Real option financing under asymmetric information. *Review of Financial Studies*, **27** (1), 180–210.

Bustamante, M. C. (2012). The dynamics of going public. *Review of Finance*, **16** (2), 577–618.

[37]Notice also that there is no relevant asymmetric information at the extremes: at date 0, both types hold the same beliefs p_0; for investments later than $\tilde{t}$, the project is internally financed. The entrepreneur has therefore the possibility to strategically select an investment date at which asymmetric information has no bite.

Décamps, J.-P. and Mariotti, T. (2004). Investment timing and learning externalities. *Journal of Economic Theory*, **118** (1), 80–102.

Dixit, A. K. and Pindyck, R. S. (1994). Investment under uncertainty, 1994. *Princeton UP, Princeton.*

Drugov, M. and Macchiavello, R. (2014). Financing experimentation. *American Economic Journal: Microeconomics*, **6** (1), 315–349.

Ewens, M., Nanda, R. and Rhodes-Kropf, M. (2014). Entrepreneurship and the cost of experimentation. *Working paper.*

Gerardi, D. and Maestri, L. (2012). A principal–agent model of sequential testing. *Theoretical Economics*, **7** (3), 425–463.

Grenadier, S. R. and Malenko, A. (2011). Real options signaling games with applications to corporate finance. *Review of Financial Studies*, **24** (12), 3993–4036.

Guedj, I. and Scharfstein, D. S. (2004). Organizational scope and investment: Evidence from the drug development strategies and performance of biopharmaceutical firms. *Working Paper.*

Halac, M., Kartik, N. and Liu, Q. (2014). Contests for experimentation. *Working paper.*

Halac, M. C., Liu, Q. and Kartik, N. (2013). Optimal contracts for experimentation. *Working paper.*

Hall, B. H. and Harhoff, D. (2012). Recent research on the economics of patents. *Working paper.*

— and Lerner, J. (2010). The financing of r&d and innovation. *Handbook of the Economics of Innovation*, **1**, 609–639.

Harhoff, D. and Stoll, S. (2014). Exploring the opaqueness of the patent system-evidence from a natural experiment. *Working Paper.*

Henkel, J. and Jell, F. (2010). Patent pending–why faster isn't always better. *Working paper.*

Hörner, J. and Samuelson, L. (2013). Incentives for experimenting agents. *The RAND Journal of Economics*, **44** (4), 632–663.

Hubbard, R. G. (1998). Capital-market imperfections and investment. *Journal of Economic Literature*, **36**, 193–225.

Keller, G., Rady, S. and Cripps, M. (2005). Strategic experimentation with exponential bandits. *Econometrica*, **73** (1), 39–68.

Kerr, W. R. and Nanda, R. (2014). Financing innovation. *Working Paper.*

Manso, G. (2011). Motivating innovation. *The Journal of Finance*, **66** (5), 1823–1860.

Morellec, E. and Schürhoff, N. (2011). Corporate investment and financing under asymmetric information. *Journal of Financial Economics*, **99** (2), 262–288.

8 Appendix

8.1 Proof of Proposition 1

Let us consider the derivative of $-\frac{1}{\lambda}\ln\frac{p_0 r(R-I)}{(1-p_0)(\lambda+r)I}$ with respect to λ :

$$\frac{1}{\lambda^2}\ln\frac{p_0 r\,(R-I)}{(1-p_0)\,(\lambda+r)\,I}+\frac{1}{\lambda\,(\lambda+r)}.$$

Its sign is given by the sign of $a(\lambda)=\ln\frac{p_0 r(R-I)}{(1-p_0)(\lambda+r)I}+\frac{\lambda}{\lambda+r}$. Note that $a'(\lambda)=-\frac{\lambda}{(\lambda+r)^2}\leq 0$, and that $\lim\limits_{\lambda\to+\infty}a(\lambda)=-\infty$. We distinguish three cases:

i) $p_0R-I<0:-\frac{1}{\lambda}\ln\frac{p_0 r(R-I)}{(1-p_0)(\lambda+r)I}>0$ for all λ, and $a(0)=\ln\frac{p_0(R-I)}{(1-p_0)I}<0$. Therefore, $t^*(\lambda)$ is positive and decreasing for all λ.

ii) $p_0R-I>0$: there exists $\lambda^*=\frac{r(p_0R-I)}{(1-p_0)I}>0$ such that $\lambda\leq\lambda^*\Leftrightarrow t^*(\lambda)=0$. Furthermore, $a(\lambda^*)=\frac{\lambda^*}{\lambda^*+r}>0$, so there also exists $\lambda^{**}>\lambda^*$ such that $a(\lambda^{**})=0$, so $t^*(\lambda)$ is increasing for $\lambda\in[\lambda^*,\lambda^{**}]$, and $t^*(\lambda)$ is decreasing for $\lambda\geq\lambda^{**}$.

iii) $p_0R-I=0$: this implies $t^*(0)=0$ and $t^*(\lambda)>0$ for all $\lambda>0$. One can show that $\lim\limits_{\lambda\to 0}\frac{1}{\lambda^2}\ln\frac{p_0 r(R-I)}{(1-p_0)(\lambda+r)I}+\frac{1}{\lambda(\lambda+r)}=+\infty$. Therefore, this case is qualitatively similar to the case $p_0R-I>0$ (t^* is increasing and then decreasing). □

8.2 D1 beliefs

In this section, we examine how D1 restricts beliefs $q(t)$. Suppose that the equilibrium prescribes that $\underline{\lambda}$ invests at $\underline{t}$ and $\overline{\lambda}$ invests at $\overline{t}$. Equilibrium beliefs $q(\underline{t})$ and $q(\overline{t})$ are pinned down by Bayes' rule, but beliefs following a deviation to $t\notin\left\{\underline{t},\overline{t}\right\}$ are not restricted and can be arbitrary. D1 imposes to attribute a deviation to date t to the type with the stronger incentive to deviate from his equilibrium action to t. Formally, let $\underline{S}(t)=\{q,W(\underline{\lambda},q,t)>W(\underline{\lambda},q(\underline{t}),\underline{t})\}$ and $\overline{S}(t)=\left\{q,W(\overline{\lambda},q,t)>W(\overline{\lambda},q(\overline{t}),\overline{t})\right\}$ the sets of out-of-equilibrium beliefs q following a deviation to t such that types $\underline{\lambda}$ and $\overline{\lambda}$ are willing to deviate from their equilibrium action to t. D1 imposes to consider $q(t)=0$ if $\overline{S}(t)\subset\underline{S}(t)$, and $q(t)=1$ if $\underline{S}(t)\subset\overline{S}(t)$.

Let $\Delta(t)\equiv W(\underline{\lambda},q,t)-W(\underline{\lambda},q(\underline{t}),\underline{t})-[W(\overline{\lambda},q,t)-W(\overline{\lambda},q(\overline{t}),\overline{t})]$ denote the difference between the marginal incentive to deviate to date t for both types, when such a deviation generates beliefs q. When $t<\tilde{t}$, we remark, using (8), that $\Delta(t)=W(\overline{\lambda},q(\overline{t}),\overline{t})-$

$W(\underline{\lambda}, q(\underline{t}), \underline{t}) - Af(t)$ is independent of q.[38]

As a consequence, $\Delta(t) > 0 \Rightarrow \overline{S}(t) \subset \underline{S}(t)$, and $\Delta(t) < 0 \Rightarrow \underline{S}(t) \subset \overline{S}(t)$. $\Delta(t) = 0 \Rightarrow \underline{S}(t) = \overline{S}(t)$, in which case we assume that $q(t) = 1$ for simplicity, but this will play no role. Overall, D1 imposes:

$$\forall t \notin \{\underline{t}, \overline{t}\} \text{ such that } t < \tilde{t},$$

$$q(t) \in \{0,1\} \text{ and } q(t) = 1 \Leftrightarrow \Delta(t) = W(\overline{\lambda}, q(\overline{t}), \overline{t}) - W(\underline{\lambda}, q(\underline{t}), \underline{t}) - Af(t) \leq 0$$

8.3 Proof of Proposition 2

It is clear that (9a) and (9c) are satisfied for $\overline{t} = \overline{t}^*$, as type $\overline{\lambda}$ gets his first best payoff. One can also show that (9d) holds for $\overline{t} = \overline{t}^*$. Suppose it does not, i.e., there is some $t_a \neq \overline{t}^*$ such that $W(\underline{\lambda}, q(t_a), t_a) > W(\underline{\lambda}, 0, \underline{t}^*)$. This implies $t_a < \tilde{t}$. Since D1 imposes to have $q(t) \in \{0,1\}$, we must have $q(t_a) = 1$, which implies:

$$\Delta(t_a) = W(\overline{\lambda}, 1, \overline{t}^*) - W(\underline{\lambda}, 0, \underline{t}^*) - Af(t_a) \leq 0.$$

Therefore, we have

$$W(\underline{\lambda}, 1, t_a) = W(\overline{\lambda}, 1, t_a) - Af(t_a) < W(\overline{\lambda}, 1, \overline{t}^*) - Af(t_a) \leq W(\underline{\lambda}, 0, \underline{t}^*).$$

A contradiction.

Consequently, (9b) is a necessary and sufficient condition for $(\underline{t}^*, \overline{t}^*)$ to be an equilibrium. Let us now derive under which condition (9b) holds at $\overline{t} = \overline{t}^*$.

Notice first that it holds if $A \geq Ie^{-r\overline{t}^*} \Leftrightarrow \overline{t}^* \geq \tilde{t}$. Indeed, we then have $W(\underline{\lambda}, 1, \overline{t}^*) = W(\underline{\lambda}, 0, \overline{t}^*) < W(\underline{\lambda}, 0, \underline{t}^*)$, so the bad type does not mimic the good type. If $A < Ie^{-r\overline{t}^*} \Leftrightarrow \overline{t}^* < \tilde{t}$, the condition becomes

$$W(\underline{\lambda}, 1, \overline{t}^*) = W(\overline{\lambda}, 1, \overline{t}^*) - Af(\overline{t}^*) \leq W(\underline{\lambda}, 0, \underline{t}^*). \tag{12}$$

It is clear that if (12) holds for some $\tilde{A}$, then it must hold for all $A > \tilde{A}$. Note also that (12) does not hold for $A = 0$. When $A \to Ie^{-r\overline{t}^*}$, $W(\underline{\lambda}, 1, \overline{t}^*) \to W(\underline{\lambda}, 0, \overline{t}^*) < W(\underline{\lambda}, 0, \underline{t}^*)$, so (12) holds. We conclude that, for any pair $(\underline{\lambda}, \overline{\lambda})$, there exists $A_0 \in (0, Ie^{-r\overline{t}^*})$ such

[38] We do not have to care about out-of-equilibrium beliefs following a deviation to $t \geq \tilde{t}$, as the bank's beliefs are irrelevant in that case.

that $(\underline{t}^*, \overline{t}^*)$ is an equilibrium iff $A \geq A_0$. In addition:

$$A_0 = \frac{W\left(\overline{\lambda}, 1, \overline{t}^*\right) - W\left(\underline{\lambda}, 0, \underline{t}^*\right)}{f(\overline{t}^*)}.$$

$\square$

8.4 Proof of Lemma 1

Consider the case $\overline{t}^* \geq t_0$.

Suppose that there is a separating equilibrium $(\underline{t}^*, \overline{t})$ such that $\overline{t}^* < \overline{t} \leq \tilde{t}$. Since $\overline{t}$ must satisfy (9b), we have

$$W(\underline{\lambda}, 0, \underline{t}^*) \geq W(\underline{\lambda}, 1, \overline{t}) = W(\overline{\lambda}, 1, \overline{t}) - Af(\overline{t}).$$

Let us denote $\alpha(\overline{t}) \equiv W(\underline{\lambda}, 0, \underline{t}^*) - W(\overline{\lambda}, 1, \overline{t}) + Af(\overline{t}) \geq 0$ the "slack" in the incentive constraint of the bad type.

One can write

$$\Delta(t) = W(\overline{\lambda}, 1, \overline{t}) - W(\underline{\lambda}, 0, \underline{t}^*) - Af(t) = A[f(\overline{t}) - f(t)] - \alpha(\overline{t}).$$

Since $\overline{t} > \overline{t}^* \geq t_0$, there exists $\epsilon > 0$ such that $\overline{t} - \epsilon \geq \overline{t}^* \geq t_0$. One has $\Delta(\overline{t} - \epsilon) < 0$, which implies $q(\overline{t} - \epsilon) = 1$. In addition, we must have $W(\overline{\lambda}, 1, \overline{t} - \epsilon) > W(\overline{\lambda}, 1, \overline{t})$ because $\overline{t} - \epsilon \geq \overline{t}^*$. Therefore, $\overline{t} - \epsilon$ is a profitable deviation for $\overline{\lambda}$, so $(\underline{t}^*, \overline{t})$ cannot be an equilibrium.

Suppose now that $\overline{t} < t_0 \leq \overline{t}^*$. By the same mechanic, one shows that type $\overline{\lambda}$ can strictly increase his payoff by deviating to $\overline{t} + \epsilon$ such that $\overline{t} + \epsilon \leq t_0 \leq \overline{t}^*$.

The proof is exactly similar in the case $t_0 \geq \overline{t}^*$. $\square$

8.5 Proof of Lemma 2

Let $h(\underline{\lambda}, \overline{\lambda}) \equiv t^*(\overline{\lambda}) - t_0(\underline{\lambda}, \overline{\lambda})$. It is easy to show that h is increasing in $\underline{\lambda}$.

In addition, $\lim\limits_{\underline{\lambda} \to 0} h(\underline{\lambda}, \overline{\lambda}) = -\infty$ and $\lim\limits_{\underline{\lambda} \to \overline{\lambda}} h(\underline{\lambda}, \overline{\lambda}) = \overline{t}^* - \frac{1}{\overline{\lambda}}$.

Therefore, we distinguish two cases:

- If $t^*(\overline{\lambda}) \leq \frac{1}{\overline{\lambda}} \Leftrightarrow \overline{\lambda} \leq \frac{e p_0 r(R-I)}{(1-p_0)I} - r$, then $t^*(\overline{\lambda}) \leq t_0$ for all $\underline{\lambda}$.
- If $t^*(\overline{\lambda}) > \frac{1}{\overline{\lambda}} \Leftrightarrow \overline{\lambda} > \frac{e p_0 r(R-I)}{(1-p_0)I} - r$, there exists a unique $\underline{\lambda}_0$ such that $h\left(\underline{\lambda}_0, \overline{\lambda}\right) = 0 \Leftrightarrow t_0(\underline{\lambda}_0, \overline{\lambda}) = t^*(\overline{\lambda})$.

Let us show that $\underline{\lambda}_0$ is a decreasing function of $\overline{\lambda}$. By the implicit function theorem,

$$\frac{\partial \underline{\lambda}_0}{\partial \overline{\lambda}} = -\frac{h_2(\underline{\lambda}_0, \overline{\lambda})}{h_1(\underline{\lambda}_0, \overline{\lambda})},$$

so $\frac{\partial \underline{\lambda}_0}{\partial \overline{\lambda}}$ has the sign of $h_2(\underline{\lambda}_0, \overline{\lambda})$.

It is easy to see that

$$h_2(\underline{\lambda}, \overline{\lambda}) = -\frac{1}{\overline{\lambda}}\left(t^*(\overline{\lambda}) - \frac{1}{\overline{\lambda}+r}\right) + \frac{\overline{\lambda}t_0 - 1}{\overline{\lambda}(\overline{\lambda}-\underline{\lambda})}$$

Since $t^*(\overline{\lambda}) = t_0$ at $\underline{\lambda}_0$, we derive

$$h_2(\underline{\lambda}_0, \overline{\lambda}) = \frac{1}{\overline{\lambda}(\overline{\lambda}-\underline{\lambda})(\overline{\lambda}+r)}\left((\overline{\lambda}+r)\underline{\lambda}t_0 - r - \underline{\lambda}\right)$$

Using $t^*(\overline{\lambda}) = t_0 < \frac{1}{\overline{\lambda}}$ at $\underline{\lambda}_0$, we derive $h_2(\underline{\lambda}_0, \overline{\lambda}) < 0$, hence the result.

Overall, we have shown

$$(\underline{\lambda}, \overline{\lambda}) \in \Lambda_H \Leftrightarrow \overline{\lambda} > \frac{ep_0 r(R-I)}{(1-p_0)I} - r \text{ and } \underline{\lambda} > \underline{\lambda}_0$$

8.6 Proof of Proposition 3

Suppose that

$$A < A_0 \Leftrightarrow W(\underline{\lambda}, 1, \overline{t}^*) > W(\underline{\lambda}, 0, \underline{t}^*).$$

Before going further, let us remark that the function $t \mapsto W(\overline{\lambda}, 1, t) - Af(t)$ is increasing on $[t_0, \overline{t}^*]$ if $t_0 < \overline{t}^*$, and decreasing on $[\overline{t}^*, t_0]$ if $\overline{t}^* < t_0$. This implies that

$$\forall t \in [\min\left(t_0, \overline{t}^*\right), \max\left(t_0, \overline{t}^*\right)],\ W(\overline{\lambda}, 1, t_0) - Af(t_0) \leq W(\overline{\lambda}, 1, t) - Af(t) \qquad (13)$$

We now establish the following lemma:

Lemma 3 *A separating equilibrium exists if and only if* $W(\underline{\lambda}, 1, t_0) \leq W(\underline{\lambda}, 0, \underline{t}^*)$.

Proof Let us first prove that $W(\underline{\lambda}, 1, t_0) \leq W(\underline{\lambda}, 0, \underline{t}^*)$ is a necessary condition for a separating equilibrium.

Suppose we have $W(\underline{\lambda}, 1, t_0) > W(\underline{\lambda}, 0, \underline{t}^*)$. This implies that $t_0 < \tilde{t}$, so one can write $W(\underline{\lambda}, 1, t_0) = W(\overline{\lambda}, 1, t_0) - Af(t_0)$.

Using (13), we derive that

$$\forall t \in \left[\min\left(t_0, \bar{t}^*\right), \max\left(t_0, \bar{t}^*\right)\right],\ W(\overline{\lambda}, 1, t) - Af(t) > W(\underline{\lambda}, 0, \underline{t}^*).$$

From Lemma 1, we derive that there is no separating equilibrium.

Let us now show that $W(\underline{\lambda}, 1, t_0) \leq W(\underline{\lambda}, 0, \underline{t}^*)$ is a sufficient condition for a separating equilibrium. Suppose $W(\underline{\lambda}, 1, t_0) \leq W(\underline{\lambda}, 0, \underline{t}^*) < W(\underline{\lambda}, 1, \bar{t}^*)$. This implies:

- If $t_0 < \bar{t}^* : \exists! \bar{t}^h \in [t_0, \bar{t}^*], W(\underline{\lambda}, 1, \bar{t}^h) = W(\underline{\lambda}, 0, \underline{t}^*)$,
- If $\bar{t}^* < t_0 \leq \tilde{t} : \exists! \bar{t}^d \in [\bar{t}^*, t_0], W(\underline{\lambda}, 1, \bar{t}^d) = W(\underline{\lambda}, 0, \underline{t}^*)$,
- If $\bar{t}^* < \tilde{t} < t_0 : \exists! \bar{t}^d \in [\bar{t}^*, \tilde{t}], W(\underline{\lambda}, 1, \bar{t}^d) = W(\underline{\lambda}, 0, \underline{t}^*)$. This is because $W(\underline{\lambda}, 1, \tilde{t}) = W(\underline{\lambda}, 0, \tilde{t}) \leq W(\underline{\lambda}, 0, \underline{t}^*)$.

Therefore, for $i \in \{h, d\}$, we always have $\bar{t}^i \leq \tilde{t}$. This implies that one can write

$$W(\underline{\lambda}, 1, \bar{t}^i) = W(\overline{\lambda}, 1, \bar{t}^i) - Af(\bar{t}^i) \text{ for } i \in \{h, d\}.$$

Let us now show that the good type has no profitable deviation starting from a candidate equilibrium $\bar{t} = \bar{t}^h$ (resp. $\bar{t}^d$), i.e. (9a) and (9c) are satisfied for $\bar{t} = \bar{t}^h$ (resp. $\bar{t}^d$).

Consider first the case $t_0 < \bar{t}^*$. Suppose the good type deviates from $\bar{t} = \bar{t}^h$ to some date t :

- If $t < \bar{t}^h$, we have $W(\overline{\lambda}, q(t), t) \leq W(\overline{\lambda}, 1, t) < W(\overline{\lambda}, 1, \bar{t}^h)$. So such a deviation cannot be profitable for any out-of-equilibrium beliefs $q(t)$.
- If $\bar{t}^h < t \leq \tilde{t}$: one can write $\Delta(t) = W(\overline{\lambda}, 1, \bar{t}^h) - W(\underline{\lambda}, 0, \underline{t}^*) - Af(t) = A[f(\bar{t}^h) - f(t)] > 0$ for $t_0 \leq \bar{t}^h < t$. Therefore, we must have $q(t) = 0$. The benefit from deviating to t becomes $W(\overline{\lambda}, 0, t) - W(\overline{\lambda}, 1, \bar{t}^h) = W(\underline{\lambda}, 0, t) + Af(t) - [W(\underline{\lambda}, 1, \bar{t}^h) + Af(\bar{t}^h)] = A[f(t) - f(\bar{t}^h)] + W(\underline{\lambda}, 0, t) - W(\underline{\lambda}, 0, \underline{t}^*) < 0$, so the benefit from this deviation is negative.
- If $\tilde{t} < t$: from $\bar{t}^* < \tilde{t}$, it is clear that such a deviation is dominated by $t = \tilde{t}$, which is not profitable thanks to the previous argument.

In the case $\bar{t}^* < t_0$, the proof is similar.

We have shown that the good type never has an incentive to deviate to an off-path investment date. To establish that he does not want to invest at $\underline{t}^*$, let us remark that, if $\overline{t}^* < \underline{t}^*$ (resp. $\overline{t}^* > \underline{t}^*$), there exists a positive (resp. negative) ϵ such that $W(\overline{\lambda}, 0, \underline{t}^*) < W(\overline{\lambda}, q(\underline{t}^* - \epsilon), \underline{t}^* - \epsilon)$ for any $q(\underline{t}^* - \epsilon)$. So deviating to $\underline{t}^*$ is always strictly dominated by some off-path deviation which has been ruled out in the previous proof.

Therefore, the good type has no profitable deviation. The last thing we need to show is that the low type cannot benefit from a deviation off path either. Suppose there exists t_a such that $W(\underline{\lambda}, q(t_a), t_a) > W(\underline{\lambda}, 0, \underline{t}^*)$. This implies that $t_a < \tilde{t}$. Since D1 imposes to have $q(t) \in \{0, 1\}$ for all t in a separating equilibrium, we must have $q(t_a) = 1$, which implies

$$\Delta(t_a) = W(\overline{\lambda}, 1, \overline{t}^i) - W(\underline{\lambda}, 0, \underline{t}^*) - Af(t_a) \leq 0 \Leftrightarrow f(\overline{t}^i) \leq f(t_a).$$

For $i = h$, $f(\overline{t}^h) \leq f(t_a) \Rightarrow W(\overline{\lambda}, 1, t_a) \leq W(\overline{\lambda}, 1, \overline{t}^h)$. Indeed, given $\overline{t}^h \geq t_0$, a necessary condition for $f(\overline{t}^h) \leq f(t_a)$ is $t_a \leq \overline{t}^h$.

Similarly, $f(\overline{t}^d) \leq f(t_a) \Rightarrow W(\overline{\lambda}, 1, t_a) \leq W(\overline{\lambda}, 1, \overline{t}^d)$ in the case $\overline{t}^* < t_0$.

Therefore, one has

$$\begin{aligned} W(\underline{\lambda}, 1, t_a) &= W(\overline{\lambda}, 1, t_a) - Af(t_a) \\ &\leq W(\overline{\lambda}, 1, \overline{t}^i) - Af(\overline{t}^i) \\ &= W(\underline{\lambda}, 1, \overline{t}^i) \\ &= W(\underline{\lambda}, 0, \underline{t}^*). \end{aligned}$$

This contradicts $W(\underline{\lambda}, 1, t_a) > W(\underline{\lambda}, 0, \underline{t}^*)$, so the bad type has no profitable deviation. □

Let us finally check under which conditions we have $W(\underline{\lambda}, 1, t_0) \leq W(\underline{\lambda}, 0, \underline{t}^*)$. First, if $t_0 \geq \tilde{t} \Leftrightarrow A \geq Ie^{-rt_0}$, we have $W(\underline{\lambda}, 1, t_0) = W(\underline{\lambda}, 0, t_0) \leq W(\underline{\lambda}, 0, \underline{t}^*)$, so this is always the case.

If $t_0 < \tilde{t}$, one can rewrite the condition as

$$W(\overline{\lambda}, 1, t_0) - Af(t_0) \leq W(\underline{\lambda}, 0, \underline{t}^*). \tag{14}$$

It is clear that, if $\tilde{A}$ satisfies (14), then so does $A > \tilde{A}$. Furthermore, (14) holds for $A = Ie^{-rt_0}$, since $W(\overline{\lambda}, 1, t_0) - Ie^{-rt_0}f(t_0) = W(\underline{\lambda}, 0, t_0) \leq W(\underline{\lambda}, 0, \underline{t}^*)$. Therefore, $\exists A_1 \leq Ie^{-rt_0}$

such that there exists a separating equilibrium if and only if $A \geq A_1$. We have:

$$A_1 = \max\left(0, \frac{W(\overline{\lambda}, 1, t_0) - W(\underline{\lambda}, 0, \underline{t}^*)}{f(t_0)}\right).$$

Finally, notice that there may be other separating equilibria involving $\bar{t} \in [t_0, \bar{t}^h)$ (resp. $\bar{t} \in (\bar{t}^d, t_0])$, but they would give a strictly lower profit to the good type than t^h (resp. t^d). $\square$

8.7 Proof of Proposition 4

Let us first establish the following lemma:

Lemma 4 *In any non-separating equilibrium, $\bar{t} = t_0$. In addition, $\underline{\lambda}$ either invests at t_0 (pooling equilibrium) or randomizes between investing at t_0 and $\underline{t}^*$ (semi-pooling equilibrium)*

Proof Let $\mathbb{T}$ be the set of dates at which both types invest with positive probability. Given that D1 imposes to consider $q(t) \in \{0, 1\}$ for each t off path, we have that $q(t) \in (0, 1) \Leftrightarrow t \in \mathbb{T}$. In addition, $t \in \mathbb{T} \Rightarrow t \leq \tilde{t}$, otherwise the good type would strictly benefit from a deviation to $\tilde{t}$.

We first establish that $\mathbb{T}$ has at most two elements. Indeed, suppose $\mathbb{T}$ has at least three distinct elements (t_a, t_b, t_c). By definition of $\mathbb{T}$, one has

$$W(\lambda, q(t_a), t_a) = W(\lambda, q(t_b), t_b) = W(\lambda, q(t_c), t_c) \text{ for } \lambda \in \{\underline{\lambda}, \overline{\lambda}\} \tag{15}$$

Using (8), one derives that $f(t_a) = f(t_b) = f(t_c)$, which is impossible, since f is continuous and single-peaked.

Suppose now that $\mathbb{T}$ has two distinct elements (t_a, t_b). One at least is different from t_0, say t_a. We then have $\Delta(t) = W(\overline{\lambda}, q(t_a), t_a) - W(\underline{\lambda}, q(t_a), t_a) - Af(t) = A[f(t_a) - f(t)]$, using (8). If $t_a < t_0$, we have $\Delta(t_a + \epsilon) < 0$, so $q(t_a + \epsilon) = 1$. Since $q(t_a) < 1$, one can always find ϵ small enough to obtain $W(\lambda, 1, t_a + \epsilon) > W(\lambda, q(t_a), t_a)$ for all λ. So there is a profitable deviation. The same reasoning holds for $t_a > t_0$.

We conclude that $\mathbb{T}$ is a singleton. If the unique element of $\mathbb{T}$ is not t_0, there is a always a profitable deviation, by the same reasoning as above. Therefore, $\mathbb{T} = \{t_0\}$.

Suppose now that $\overline{\lambda}$ invests with positive probability at some $t_a \neq t_0$, with $t_a \leq \tilde{t}$. Since $\mathbb{T} = \{t_0\}$, $\underline{\lambda}$ must then invest with probability zero at date t_a.

We therefore have

$$W(\overline{\lambda}, 1, t_a) = W(\overline{\lambda}, q(t_0), t_0) = W(\underline{\lambda}, q(t_0), t_0) + Af(t_0).$$

This implies that

$$W(\underline{\lambda}, 1, t_a) = W(\overline{\lambda}, 1, t_a) - Af(t_a) = W(\underline{\lambda}, q(t_0), t_0) + A[f(t_0) - f(t_a)] > W(\underline{\lambda}, q(t_0), t_0).$$

So type $\underline{\lambda}$ then strictly prefers to invest at date t_a than at t_0, which is impossible. $\square$

Let us now turn to the proof of the Proposition. Suppose first that a separating equilibrium exists, i.e., $A \geq A_1$. Using Lemma 3, this is equivalent to $W(\underline{\lambda}, 1, t_0) \leq W(\underline{\lambda}, 0, \underline{t}^*)$. We then have:

$$W(\underline{\lambda}, q(t_0), t_0) < W(\underline{\lambda}, 1, t_0) \leq W(\underline{\lambda}, 0, \underline{t}^*)$$

Therefore, the bad type cannot invest at t_0 with positive probability, as this is strictly dominated by investing at $\underline{t}^*$. Therefore, there is neither semi-pooling nor pooling equilibria.

Before we derive equilibrium conditions for a pooling and a semi-pooling equilibrium, notice that, since the equilibrium payoffs are $W(\lambda, q(t_0), t_0)$ for each type λ, we have $\Delta(t) = A[f(t_0) - f(t)] > 0$, so any off-path deviation generates beliefs $q(t) = 0$.

Conditions for a pooling equilibrium The following conditions must be satisfied for a pooling equilibrium $\underline{t} = \overline{t} = t_0$ to exist:

$$W(\underline{\lambda}, q_0, t_0) \geq W(\underline{\lambda}, 0, \underline{t}^*) \tag{16}$$

$$W(\overline{\lambda}, q_0, t_0) \geq W(\overline{\lambda}, 0, t) \text{ for all } t \leq \tilde{t}. \tag{17}$$

A necessary condition for a pooling equilibrium is that a separating equilibrium does not exist, i.e., $W(\underline{\lambda}, 1, t_0) > W(\underline{\lambda}, 0, \underline{t}^*)$. Since $W(\underline{\lambda}, q, t_0)$ is increasing in q, and since $W(\underline{\lambda}, 0, t_0) \leq W(\underline{\lambda}, 0, \underline{t}^*)$, there exists a critical value of $\overline{q}$ such that (16) holds if and only if $q_0 \geq \overline{q}$. $\overline{q}$ satisfies

$$W(\underline{\lambda}, \overline{q}, t_0) = W(\underline{\lambda}, 0, \underline{t}^*).$$

Let $\hat{t} \in \arg\max_t W(\overline{\lambda}, 0, t)$. First, note that we must have $\hat{t} \leq \tilde{t}$. Otherwise, the good type could increase his payoff by slightly reducing t below $\hat{t}$: the bank's beliefs are irrelevant in this range, and this increases his profit, since $\overline{t}^* < \tilde{t}$. Therefore, one may write:

$$\begin{aligned} W(\overline{\lambda}, \overline{q}, t_0) - W(\overline{\lambda}, 0, \hat{t}) &= W(\underline{\lambda}, \overline{q}, t_0) + Af(t_0) - W(\underline{\lambda}, 0, \hat{t}) - Af(\hat{t}) \\ &= W(\underline{\lambda}, 0, \underline{t}^*) - W(\underline{\lambda}, 0, \hat{t}) + A\left[f(t_0) - f(\hat{t})\right] \end{aligned}$$

This term is positive, as f is maximized at t_0, and by definition of $\underline{t}^*$. This implies that (16) $\Rightarrow$ (17). Consequently, there is a pooling equilibrium in which all types invest at $t = t_0$ if and only if $q_0 \geq \overline{q}$. To derive this equilibrium condition as a function of A, let us first notice that, at $A = A_1$, we have $\overline{q} = 1 > q_0$, by definition of A_1. So, the pooling equilibrium does not exist when A is sufficiently close to A_1. Furthermore, one can rewrite (7) as

$$W(\underline{\lambda}, q, t) = W(\underline{\lambda}, 0, t) + qf(t)[Ie^{-rt} - A]$$

to see that

$$\overline{q} = \frac{W(\underline{\lambda}, 0, \underline{t}^*) - W(\underline{\lambda}, 0, t_0)}{f(t_0)[Ie^{-rt_0} - A]}$$

Therefore, $\overline{q}$ is increasing in A. When $A = 0$, we have $\overline{q} = \frac{W(\underline{\lambda},0,\underline{t}^*) - W(\underline{\lambda},0,t_0)}{f(t_0)Ie^{-rt_0}}$. We derive that:

- if $q_0 \leq \frac{W(\underline{\lambda},0,\underline{t}^*) - W(\underline{\lambda},0,t_0)}{Ie^{-rt_0}f(t_0)}$, no pooling equilibrium ever exists,
- if $q_0 > \frac{W(\underline{\lambda},0,\underline{t}^*) - W(\underline{\lambda},0,t_0)}{Ie^{-rt_0}f(t_0)}$, a pooling equilibrium exists if and only if A is small enough.

Overall, a pooling equilibrium exists if and only if $A \leq A_2$, with

$$A_2 = \max\left(0, Ie^{-rt_0} - \frac{W(\underline{\lambda}, 0, \underline{t}^*) - W(\underline{\lambda}, 0, t_0)}{q_0 f(t_0)}\right).$$

Conditions for a semi-pooling equilibrium The following conditions must be satisfied for a semi-pooling equilibrium to exist:

$$W(\underline{\lambda}, q(t_0), t_0) = W(\underline{\lambda}, 0, \underline{t}^*) \tag{18}$$

$$W(\overline{\lambda}, q(t_0), t_0) > W(\overline{\lambda}, 0, t) \text{ for all } t. \tag{19}$$

Using the same argument as above, it is easy to see that (18) $\Rightarrow$ (19).

Finally, it is obvious that $q(t_0) > q_0$, because $\underline{\lambda}$ does not invest at t_0 with probability 1, whereas $\overline{\lambda}$ does. So, if $q_0 \geq \overline{q} \Leftrightarrow A \leq A_2$, (18) cannot hold. Conversely, if $q_0 < \overline{q}$, type $\underline{\lambda}$ can always invest at date t_0 with a probability $x \in (0,1)$ such that $q(t_0) = \frac{q_0}{q_0+(1-q_0)x} = \overline{q}$, in which case (18) is satisfied. Hence the result. □

8.8 Proof of Proposition 5

If $A \in [A_1, A_0)$, in the least cost separating equilibrium, the good type invests at date $\overline{t}^i$ given by

$$W(\underline{\lambda}, 1, \overline{t}^i) = W(\overline{\lambda}, 1, \overline{t}^i) - Af(\overline{t}^i) = W(\underline{\lambda}, 0, \underline{t}^*). \tag{20}$$

Differentiating with respect to A yields

$$\frac{\partial \overline{t}^i}{\partial A} = \frac{f(\overline{t}^i)}{W_3\left(\overline{\lambda}, 1, \overline{t}^i\right) - A\frac{\partial f}{\partial t}(\overline{t}^i)}.$$

The numerator is positive. The denominator is negative for $i = d$, as $\overline{t}^d \in \left[\overline{t}^*, t_0\right]$, and positive for $i = h$, since $\overline{t}^h \in \left[t_0, \overline{t}^*\right]$. This implies $\frac{\partial \overline{t}^h}{\partial A} > 0$, and $\frac{\partial \overline{t}^d}{\partial A} < 0$. □

8.9 Proof of Corollary 4

Using (7), one rewrites

$$\begin{aligned} W(\underline{\lambda}, q, t) &= W(\underline{\lambda}, 0, t) + qf(t)[Ie^{-rt} - A], \\ W(\overline{\lambda}, q, t) &= W(\overline{\lambda}, 1, t) - (1-q)f(t)[Ie^{-rt} - A]. \end{aligned}$$

i) If $A < A_2$, the equilibrium involves pooling at t_0. Using the above relationships, we derive that $W(\overline{\lambda}, q_0, t_0)$ is increasing in A, while $W(\underline{\lambda}, q_0, t_0)$ is decreasing in A. The expected welfare in this equilibrium reads

$$q_0 W(\overline{\lambda}, q_0, t_0) + (1-q_0)W(\underline{\lambda}, q_0, t_0) = q_0 W(\overline{\lambda}, 1, t_0) + (1-q_0)W(\underline{\lambda}, 0, t_0),$$

hence is independent of A.

ii) If $A \geq A_2$, the equilibrium payoff of the bad type is constant and equal to $W(\underline{\lambda}, 0, \underline{t}^*)$, so we only need to focus on the payoff of the good type.

- If $A_2 \leq A \leq A_1$, this payoff equals

$$\begin{aligned} W(\overline{\lambda}, \overline{q}, t_0) &= W(\underline{\lambda}, \overline{q}, t_0) + Af(t_0) \\ &= W(\underline{\lambda}, 0, \underline{t}^*) + Af(t_0), \end{aligned}$$

  so the equilibrium payoff of the good type linearly increases in A.

- If $A_1 < A < A_0$, the good type gets $W(\overline{\lambda}, 1, \overline{t}^i)$, with $i \in \{h, d\}$.

  $\frac{\partial W}{\partial A}(\overline{\lambda}, 1, \overline{t}) = W_3\left(\overline{\lambda}, 1, \overline{t}^i\right) \frac{\partial \overline{t}^i}{\partial A} > 0$ for $i = \{h, d\}$, using Proposition 5.

- If $A \geq A_0$, the good types reaches his first best profit, which is independent of A.

Finally, the equilibrium payments are both continuous in A. To see this, it is enough to notice that, if $A_2 > 0$, one has $\lim\limits_{A \to A_2^+} \overline{q} = q_0$. In addition, one has $\lim\limits_{A \to A_1^+} \overline{q} = 1$, and $\lim\limits_{A \to A_0^-} \overline{t}^i = \overline{t}^*$, so continuity obtains everywhere. □

8.10 Proof of the existence of reversals

8.10.1 Timing reversal

Suppose that $\overline{t}^* < \underline{t}^* < t_0 < \tilde{t}$, and that the first best is not an equilibrium, that is, $W(\underline{\lambda}, 1, \overline{t}^*) > W(\underline{\lambda}, 0, \underline{t}^*)$. Since $W(\underline{\lambda}, 1, t)$ is decreasing in t on $[\overline{t}^*, t_0]$, and since $W(\underline{\lambda}, 1, \underline{t}^*) > W(\underline{\lambda}, 0, \underline{t}^*)$, a separating equilibrium, if it exists (that is, $A \geq A_1$) must be such that $\overline{t} > \underline{t}^*$, hence the reversal.

It is easy to show that $\overline{t}^* = \underline{t}^* \Rightarrow \overline{t}^* = \underline{t}^* < t_0$. Therefore, there are generically parameters values such that $\overline{t}^* < \underline{t}^* < t_0$, independently of A. In addition, $A_1 < Ie^{-rt_0}$, so for A in $[A_1, Ie^{-rt_0}]$, we have both $t_0 < \tilde{t}$ and the existence of a separating equilibrium.

8.10.2 Belief reversal

It is obvious to see that $p^*(\overline{\lambda}, \overline{t}) - p^*(\underline{\lambda}, \underline{t}^*)$ has the same sign as $e^{-\underline{\lambda t}^*} - e^{\overline{\lambda t}}$.

If $A \geq A_0$, this difference reads $e^{-\underline{\lambda t}^*} - e^{-\overline{\lambda t}^*} = \frac{rp_0(R-I)}{(1-p_0)I}\left(\frac{1}{\underline{\lambda}+r} - \frac{1}{\overline{\lambda}+r}\right) > 0$.

So the good type is always more optimistic in this case.

Suppose now that $A \in [A_1, A_0]$. If $(\underline{\lambda}, \overline{\lambda}) \in \Lambda_D$, $\overline{t} > \overline{t}^*$, so the good type is more optimistic than in the first best upon investing, so he is a fortiori more confident than the bad type.

To show that belief reversal is possible, let us consider $\underline{\lambda} > \frac{ep_0 r(R-I)}{(1-p_0)I} - r$. This implies that, for any $\overline{\lambda}$, $(\underline{\lambda}, \overline{\lambda}) \in \Lambda_H$.[39]

In this case, $\overline{t}$ is increasing in A, meaning that $e^{-\underline{\lambda t}^*} - e^{-\overline{\lambda}\overline{t}}$ is increasing in A as well. Furthermore, when $A \to A_0$, the difference must be positive, as $\overline{t} \to \overline{t}^*$.

When $A \to A_1$, $\overline{t}$ tends to t_0. Let us show that $e^{-\underline{\lambda t}^*} - e^{-\overline{\lambda}t_0}$ may take negative values.

To see this, let us observe that the derivative of $\overline{\lambda}t_0$ with respect to $\overline{\lambda}$ has the sign of $\overline{\lambda} - \underline{\lambda} - \underline{\lambda}\left(\ln \overline{\lambda} - \ln \underline{\lambda}\right)$, which is nonnegative. So $e^{-\underline{\lambda t}^*} - e^{-\overline{\lambda}t_0}$ is nondecreasing in $\overline{\lambda}$. Using $\lim_{\overline{\lambda} \to \underline{\lambda}} t_0 = \frac{1}{\underline{\lambda}}$, we derive that

$$\lim_{\overline{\lambda} \to \underline{\lambda}} e^{-\underline{\lambda t}^*} - e^{\overline{\lambda}t_0} = \frac{rp_0(R-I)}{(1-p_0)(\underline{\lambda}+r)I} - \frac{1}{e}.$$

This limit is negative, as we have assumed $\underline{\lambda} > \frac{ep_0 r(R-I)}{(1-p_0)I} - r$.

We conclude that, for A and $\overline{\lambda}$ sufficiently small, $e^{-\underline{\lambda t}^*} - e^{\overline{\lambda}\overline{t}} < 0$, meaning that the bad type is more confident upon investing than the good type.

8.11 Proof of Proposition 6

Let us now establish how $\overline{t}^h$ and $\overline{t}^d$ vary with $\overline{\lambda}$.

Differentiating (20) with respect to $\overline{\lambda}$ gives

$$\frac{\partial \overline{t}^i}{\partial \overline{\lambda}} = \frac{(1-p_0)\overline{t}^i e^{-\overline{\lambda}\overline{t}^i}\left(A - e^{-r\overline{t}^i} I\right)}{W_3\left(\overline{\lambda}, 1, \overline{t}^i\right) - A\frac{\partial f}{\partial t}(\overline{t}^i)}.$$

The denominator is negative for $i = d$, and positive for $i = h$; the numerator is negative since $\overline{t}^i < \widetilde{t}$, hence the result.

8.12 Proof of Proposition 7

Recall that

$$\overline{t}^* = \max\left(-\frac{1}{\overline{\lambda}} \ln \frac{p_0 r\,(R-I)}{(1-p_0)\left(\overline{\lambda}+r\right) I}, 0\right).$$

Let $p^* = \frac{(\overline{\lambda}+r)I}{rR+\overline{\lambda}I}$ be smallest value such that $\overline{t}^* = 0$. Notice that $p^* > \frac{I}{R}$.

We have

$$\frac{\partial \overline{t}^*}{\partial p_0} = -\frac{1}{\overline{\lambda}p_0(1-p_0)} \text{ for } p_0 < p^* \tag{21}$$

[39]This is because $\underline{\lambda}_0(\frac{ep_0 r(R-I)}{(1-p_0)I} - r) = \frac{ep_0 r(R-I)}{(1-p_0)I} - r$ and $\underline{\lambda}_0(\overline{\lambda})$ is decreasing in $\overline{\lambda}$.

Let p^{**} be such that $\overline{t}^*|_{p_0=p^{**}} = \tilde{t}$. We have $p^{**} < p^*$.

Let us also define $t_1 \equiv \overline{t}^*|_{p_0 R=I} = -\frac{1}{\overline{\lambda}} \ln \frac{r}{\overline{\lambda}+r}$ the optimal investment date when $p_0 R - I = 0$.

Finally, let us define the following function:

$$g(p_0) \equiv W(0, 1, \overline{t}^*) - W(0, 0, \underline{t}^*)$$

Since $W(0, 0, \underline{t}^*) = 0$ if $p_0 R < I$ and $W(0, 0, \underline{t}^*) = p_0 R - I$ if $p_0 R \geq I$, g is continuous in p_0. In addition, we know that the first best exists whenever p_0 is such that $g(p_0) \leq 0$.

- If $p_0 \leq p^{**}, \overline{t}^* \geq \tilde{t} \Rightarrow g(p_0) = W(0, 0, \overline{t}^*) - W(0, 0, \underline{t}^*) \leq 0$, so the first best exists.

- If $p_0 \geq p^*, g(p_0) = 0$, so the first best also exists.

Therefore, we only need to study g on $[p^{**}, p^*]$. Whenever $p_0 > p^{**}$, one may rewrite

$$g(p_0) = W(\overline{\lambda}, 1, \overline{t}^*) - A(1 - p_0)(1 - e^{-\overline{\lambda}\overline{t}^*}) - W(0, 0, \underline{t}^*).$$

We distinguish two cases:

- If $p^{**} \geq \frac{I}{R} \Leftrightarrow t_1 \geq \tilde{t} \Leftrightarrow A \geq I(\frac{r}{\overline{\lambda}+r})^{r/\overline{\lambda}}$: For all $p_0 \in [p^{**}, p^*]$, we derive, using the envelope theorem:

$$g'(p_0) = e^{-r\overline{t}^*}(R - I + Ie^{-\overline{\lambda}\overline{t}^*}) + A(1 - e^{-\overline{\lambda}\overline{t}^*}) - A(1 - p_0)\overline{\lambda}\frac{\partial \overline{t}^*}{\partial p_0}e^{-\overline{\lambda}\overline{t}^*} - R$$

  Using (21), we rewrite $g'(p_0) = e^{-r\overline{t}^*}(R - I + Ie^{-\overline{\lambda}\overline{t}^*}) + A\frac{p_0+(1-p_0)e^{-\overline{\lambda}\overline{t}^*}}{1-p_0} - R$

  Replacing $e^{-\overline{\lambda}\overline{t}^*}$, we conclude $g'(p_0) = e^{-r\overline{t}^*}(R - I + Ie^{-\overline{\lambda}\overline{t}^*}) + A\frac{rR+\overline{\lambda}I}{(\overline{\lambda}+r)I} - R$.

  It is easy to see that g' is increasing in p_0, so g is convex. Since $g(p^{**}) \leq 0$ and $g(p^*) = 0$, we derive that $g(p_0) \leq 0$ for all p_0. Therefore, the first best is always an equilibrium.

- If $p^{**} < \frac{I}{R} \Leftrightarrow t_1 < \tilde{t} \Leftrightarrow A < I(\frac{r}{\overline{\lambda}+r})^{r/\overline{\lambda}}$, we need to distinguish two regions:

  - $p_0 \in [p^{**}, \frac{I}{R}]$: On this interval, one has

$$g'(p_0) = e^{-r\overline{t}^*}(R - I + Ie^{-\overline{\lambda}\overline{t}^*}) + A\frac{rR + \overline{\lambda}I}{(\overline{\lambda} + r)I} > 0$$

    In addition, $g(p^{**}) \leq 0$ and $g(\frac{I}{R}) = \frac{(R-I)}{R}(1 - e^{-\overline{\lambda}t_1})(Ie^{-r\overline{t}_1} - A) > 0$.

    Therefore, there exists $\underline{p} \in [p^{**}, \frac{I}{R})$ such that $g(p_0) \leq 0 \Leftrightarrow p_0 \leq \underline{p}$.

– $p_0 \in [\frac{I}{R}, p^*]$: We then have $g'(p^*) = A\frac{rR+\overline{\lambda}I}{(\overline{\lambda}+r)I} > 0$. Recalling that g is convex on $[\frac{I}{R}, p^*]$, and given that $g(\frac{I}{R}) > 0$ by continuity at $p_0 = \frac{I}{R}$, and that $g(p^*) = 0$, we conclude that there exists $\overline{p} \in (\frac{I}{R}, p^*]$ such that $g(p_0) \leq 0 \Leftrightarrow p_0 \geq \overline{p}$.

We conclude that the first best is an equilibrium if and only if $p_0 \notin [\underline{p}, \overline{p}]$.

Let us now suppose that the first best is not an equilibrium. Then, the equilibrium involves separation at $\overline{t}^d$ such that

$$W(\overline{\lambda}, 1, \overline{t}^d) - A(1-p_0)(1-e^{-\overline{\lambda}\overline{t}^d}) = \max\{p_0 R - I, 0\}.$$

$$\Leftrightarrow e^{-r\overline{t}^d}(p_0(R-I) - (1-p_0)e^{-\overline{\lambda}\overline{t}^d}I) - A(1-p_0)(1-e^{-\overline{\lambda}\overline{t}^d}) = \max\{p_0 R - I, 0\}. \quad (22)$$

Let $d \equiv \frac{\partial}{\partial t}W(\overline{\lambda}, 1, t) - \frac{\partial}{\partial t}A(1-p_0)(1-e^{-\overline{\lambda}t}) < 0$ denote the derivative of the left handside with respect to t. It is negative since the equilibrium is delayed.

Let us start with the case $p_0 R - I < 0$. One has:

- $\frac{\partial t^d}{\partial p_0} = -\frac{1}{d}[A(1-e^{-\overline{\lambda}\overline{t}^d}) + e^{-r\overline{t}^d}(R - I + Ie^{-\overline{\lambda}\overline{t}^d})] > 0$
- $\frac{\partial t^d}{\partial R} = -\frac{1}{d}p_0 e^{-r\overline{t}^d} > 0$
- $\frac{\partial t^d}{\partial I} = \frac{1}{d}e^{-r\overline{t}^d}(p_0 + (1-p_0)e^{-\overline{\lambda}\overline{t}^d}) < 0$.

Let us now consider the case $p_0 R - I \geq 0$. One has:

- $\frac{\partial t^d}{\partial p_0} = -\frac{1}{d}[A(1-e^{-\overline{\lambda}\overline{t}^d}) + e^{-r\overline{t}^d}(R - I + Ie^{-\overline{\lambda}\overline{t}^d}) - R]$

  This has the sign of

$$A(1-p_0)(1-e^{-\overline{\lambda}\overline{t}^d}) + (1-p_0)e^{-r\overline{t}^d}(R - I + Ie^{-\overline{\lambda}\overline{t}^d}) - (1-p_0)R$$

$$= e^{-r\overline{t}^d}\left(p_0(R-I) - (1-p_0)e^{-\overline{\lambda}\overline{t}^d}I\right) - (p_0R - I) + (1-p_0)e^{-r\overline{t}^d}(R - I + Ie^{-\overline{\lambda}\overline{t}^d}) - (1-p_0)R$$

$$= -(1-e^{-r\overline{t}^d})(R-I) < 0,$$

  where the second inequality makes use of (22).

- $\frac{\partial t^d}{\partial R} = -\frac{1}{d}p_0(e^{-r\overline{t}^d} - 1) < 0$
- $\frac{\partial t^d}{\partial I} = -\frac{1}{d}\left(1 - e^{-r\overline{t}^d}(p_0 + (1-p_0)e^{-\overline{\lambda}\overline{t}^d})\right) > 0$.

□